String theory flux vacua on twisted tori and Generalized Complex Geometry

David Andriot

LABORATOIRE DE PHYSIQUE THEORIQUE ET HAUTES ENERGIES

THESE DE DOCTORAT DE

L'UNIVERSITE PIERRE ET MARIE CURIE PARIS 6

Spécialité: **Physique**

Présentée par

David ANDRIOT

Pour obtenir le grade de

DOCTEUR de l'UNIVERSITE PIERRE ET MARIE CURIE PARIS 6

Sujet:

STRING THEORY FLUX VACUA ON TWISTED TORI AND GENERALIZED COMPLEX GEOMETRY

SOLUTIONS AVEC FLUX DE LA THEORIE DES CORDES SUR TORES TWISTES, ET GEOMETRIE COMPLEXE GENERALISEE

Soutenue le 1er juillet 2010

devant le jury composé de

M.	Iosif BENA,	examinateur,
Mme	Michela PETRINI,	directrice de thèse,
M.	Henning SAMTLEBEN,	rapporteur,
M.	Dimitrios TSIMPIS,	rapporteur,
M.	Daniel WALDRAM,	examinateur,
M.	Jean-Bernard ZUBER,	examinateur.

Remerciements

Je remercie en premier lieu ma directrice de thèse Michela Petrini. Je la remercie tout d'abord d'avoir accepté de me prendre comme étudiant de thèse. Elle m'a ainsi permis d'entrer dans le monde de la recherche, ce dont je rêvais depuis longtemps. Elle m'a également introduit à une thématique de recherche qui s'est avérée me plaire énormément. Je la remercie ensuite d'avoir supporté certains de mes travers, notamment mon style rédactionnel et mes goûts cinématographiques. Ceux-ci n'ont pas toujours été adaptés à sa notion de l'esthétique, qui restera pour moi un idéal. Plus sérieusement, je la remercie d'avoir su accepter mon fort besoin d'indépendance. Elle m'a laissé me développer (pour reprendre ses mots, "j'ai appris à faire mes lacets" tout seul), tout en me recadrant à chaque fois que cela était nécessaire. Elle m'a de plus toujours dirigé vers les bonnes personnes, et toujours soutenu quand cela était nécessaire. Son sens professionnel m'a été très profitable. Je la remercie aussi pour son apport scientifique majeur. Son sens de la physique, son haut niveau scientifique et sa connaissance générale de la théorie des cordes sont incontestables. J'espère pouvoir continuer à en profiter par la suite. Enfin, je la remercie pour son naturel très humain, sa gentillesse, et ses qualités sociales, qui ont permis une base très agréable dans nos relations.

Comment ne pas remercier ensuite celui qui a joué également un rôle prédominant, mon collaborateur Ruben Minasian. Je le remercie avant tout pour ses nombreux conseils non-scientifiques, son soutien dans de nombreuses situations, et son sens social. Ils ont été déterminants durant ma thèse et pour la suite. Son apport scientifique a aussi été fondamental, en particulier en géométrie différentielle. Il a également su collaborer de manière constructive, tout en respectant nos différentes manières de procéder. Je le remercie enfin pour ses idées tantôt folles ou brillantes, pour son écoute, pour ses multiples anecdotes, et pour son humour grinçant.

Je remercie enfin mon compagnon d'infortune, avec qui j'ai souffert dans les déserts mathématiques des algèbres résolubles, dans certains calculs effrayants et interminables, et dans le quotidien du travail de recherche: mon collaborateur Enrico Goi. Ses ressources mathématiques, sa disponibilité, et sa bonne volonté ont certainement été d'un grand soutien.

Je remercie ensuite toutes les personnes qui ont eu le courage, la curiosité ou le devoir d'assister à ma soutenance de thèse. J'espère qu'au moins le pot de thèse les récompensera. Je remercie en particulier les membres de mon jury, Iosif Bena, Henning Samtleben, Dimitrios Tsimpis, Daniel Waldram, et Jean-Bernard Zuber, pour avoir accepté de participer, pour le temps accordé, et les remarques pertinentes qu'ils m'ont fournies. Je remercie expressément mes deux rapporteurs, Henning Samtleben et Dimitrios Tsimpis, pour avoir accepté ce travail, et l'avoir fait si efficacement et de manière constructive.

Je remercie les membres du LPTHE pour l'accueil chaleureux et l'ouverture scientifique qui règne dans le laboratoire. Ce cadre de travail a été non seulement agréable mais stimulant. Je remercie en particulier le personnel administratif, à savoir Olivier Babelon, Damien Brémont, Françoise Got, Isabelle Nicolaï, Lionel Pihery, Annie Richard, Valérie Sabouraud, et tous les autres, pour avoir géré toutes ces choses que nous ignorons, mais également tous mes voyages, mes conférences, mes divers problèmes, de manière toujours efficace.

Je remercie également le groupe de théorie des cordes du LPTHE. Son dynamisme, son ouverture, sa bonne humeur m'ont énormément apporté. Je remercie plus largement le groupe de théorie des cordes de la région parisienne. Là encore, l'activité, le dynamisme, la facilité avec laquelle il était possible de se rencontrer, discuter, m'a offert un cadre extrêmement enrichissant pour ma thèse.

Durant ma thèse, j'ai également beaucoup voyagé, et cela m'a été très profitable. Je remercie toutes les personnes et les organismes qui ont rendu cela possible, notamment le LPTHE, le LPTENS, la FRIF, le CNRS, l'UPMC, l'ANR, ainsi que tous les laboratoires que j'ai visité. Je remercie les organisateurs des écoles *Paris - Bruxelles - Amsterdam 2006*, *RTN Winter School 2008* au CERN, et des écoles de mai 2008 et de mars 2009 à l'ICTP à Trieste. Ces écoles ont été l'occasion pour moi d'apprendre énormément et de faire de nombreuses rencontres. Je remercie tous ceux qui y ont participé, étudiants ou chercheurs, pour avoir rendu ces moments enrichissants et agréables. J'ai également assisté voire participé à quelques conférences dont le *4th EU RTN Workshop* à Varna. Je remercie les organisateurs pour les mêmes raisons. Enfin, j'ai effectué de nombreuses visites de laboratoires et donné des séminaires, aux Etats-Unis et en Europe. Je remercie toutes les personnes qui m'y ont invité, toutes celles qui m'ont reçu et rendu mes séjours plaisants. Les échanges scientifiques que j'ai pu avoir dans ces occasions, mais également mon aperçu de la communauté internationale, m'ont été très précieux.

Je remercie enfin tous les scientifiques avec qui j'ai pu échanger au cours de ma thèse, en particulier: Ch. Bock, Agostino Butti, Davide Cassani, Atish Dabholkar, Mark Goodsell, Mariana Graña, Amir Kashani-Poor, Paul Koerber, Guiseppe Policastro, Waldemar Schulgin, Ashoke Sen, Alessandro Tomasiello, Ian Troost, Dimitrios Tsimpis, Alberto Zaffaroni, etc. Un remerciement particulier à Gregory Korchemsky, qui a toujours été de bon conseil avant, puis pendant ma thèse. Je le remercie également pour son apport scientifique important, en particulier en théorie des champs, et pour son soutien solide pour mon après-thèse.

Plus personnellement, je remercie tous mes collègues étudiants du LPTHE. Plus que d'avoir rendu ma thèse agréable, j'ai passé avec eux des moments inoubliables. Merci à Camille Aron, Guillaume Bossard, Solène Chevalier Thery, Quentin Duret, Benoît Estienne, Tiago Fonseca, Joao Gomes, Nicolas Houdeau, Demian Levis, Ann-Kathrin Maier, Alexis Martin, Yann Michel, César Moura, Redamy Perez Ramos, Clément Rousset, Mathieu Rubin, Alberto Sicilia, et leurs moitiés. Sans oublier Bruno Machet. Merci à vous pour les time challenges dont je fus un fervent adepte, merci pour les cake challenges, les records de Pacman, les têtes à claques, la chanson du dimanche, les repas thésards, les déjeuners à la cantine, etc. Un merci particulier au groupe du Brésil avec qui j'ai réalisé un voyage extraordinaire, et bien plus par ailleurs. Enfin, une pensée toute particulière pour Mathieu Rubin, avec qui j'ai partagé mon bureau pendant quatre ans. Il a subi mon humeur, mes insultes, mon humour discutable, mes demandes de service répétées, les discussions avec mes collaborateurs, mes coups de téléphones, etc. Il est néanmoins parvenu à faire une excellente thèse, ce dont je ne pouvais douter. Merci pour son aide informatique, et mathématique, nos discussions m'auront souvent aidé à avancer, ne serait-ce que dans ma recherche. Sa présence et son caractère inimitable me manqueront.

Un grand merci à ma famille et à mes amis qui m'ont soutenu à leur manière durant ces quatre années. Ils m'ont souvent permis de garder les pieds sur terre, et ont su respecter la voie que j'avais choisie. En particulier, merci à mes parents et ma famille pour m'avoir permis et encouragé à me lancer dans cette voie pour le moins hasardeuse. Merci également à tous pour les efforts de vulgarisation que vous m'avez demandés, qui m'ont permis de rester en phase avec le monde réel. Merci également à Marc Lilley avec qui j'ai beaucoup échangé scientifiquement et personnellement, en particulier à l'occasion de la création de la conférence SCSC09, mais pas seulement. Enfin, un merci particulier à mon ami et collègue Benjamin Basso avec qui j'ai partagé de nombreux doutes, des questions de physique, des moments de lassitude comme de joies scientifiques, et bien plus que cela. Une telle relation est rare et privilégiée. J'espère préserver celle-ci aussi longtemps que possible.

Tous ces remerciements ne sont rien à côté de ceux dûs à Cécile. En quelques mots, forcément insuffisants: merci pour ta présence, ton écoute, ta patience, ton soutien, permanents pendant ces quatre années. Merci d'avoir accepté tant bien que mal toutes mes heures de travail, et les conséquences que cela peut avoir. Merci pour ta joie de vivre qui a su si bien me changer les idées quand j'en avais besoin. Merci de m'avoir aimé malgré tous les travers auxquels ce métier peut mener. Merci d'être telle que tu es, tu m'as fourni un équilibre sans lequel je serais certainement tombé depuis longtemps.

Abstract

This thesis is devoted to the study of flux vacua of string theory, with the ten-dimensional space-time split into a four-dimensional maximally symmetric space-time, and a six-dimensional internal manifold M, taken to be a solvmanifold (twisted torus). Such vacua are of particular interest when trying to relate string theory to supersymmetric (SUSY) extensions of the standard model of particles, or to cosmological models.

For SUSY solutions of type II supergravities, allowing for fluxes on M helps to solve the moduli problem. Then, a broader class of manifolds than just the Calabi-Yau can be considered for M, and a general characterization is given in terms of Generalized Complex Geometry: M has to be a Generalized Calabi-Yau (GCY).

A subclass of solvmanifolds have been proven to be GCY, so we look for solutions with such M. To do so, we use an algorithmic resolution method. Then we focus on specific new solutions: those admitting an intermediate $SU(2)$ structure.

A transformation named the twist is then discussed. It relates solutions on torus to solutions on solvmanifolds. Working out constraints on the twist to generate solutions, we can relate known solutions, and find a new one. We also use the twist to relate flux vacua of heterotic string.

Finally we consider ten-dimensional de Sitter solutions. Looking for such solutions is difficult, because of several problems among which the breaking of SUSY. We propose an ansatz for SUSY breaking sources which helps to overcome these difficulties. We give an explicit solution on a solvmanifold, and discuss partially its four-dimensional stability.

A long French summary of the thesis can be found in appendix D.

Résumé court

Nous étudions des solutions avec flux de la théorie des cordes, sur un espace-temps dix-dimensionnel séparé en un espace-temps quatre-dimensionnel maximalement symétrique, et une variété interne six-dimensionnelle M, étant ici une variété résoluble (un tore twisté). Ces solutions sont intéressantes pour relier la théorie des cordes à une extension supersymétrique (SUSY) du modèle standard des particules, ou à des modèles cosmologiques.

Pour des solutions SUSY des supergravités de type II, la présence de flux sur M aide à résoudre le problème des moduli. Une classe plus large de variétés que le simple Calabi-Yau peut alors être considérée pour M, et une caractérisation générale est donnée en terme de Géométrie Complexe Généralisée: M doit être un Calabi-Yau Généralisé (GCY).

Il a été montré qu'une sous-classe de variétés résolubles sont des GCY, donc nous allons chercher des solutions sur de telles M. Pour y parvenir, nous utilisons une méthode de résolution algorithmique. Nous étudions ensuite un certain type de solutions: celles qui admettent une structure $SU(2)$ intermédiaire.

Par la suite, nous considérons le twist, une transformation qui relie des solutions sur le tore à d'autres sur variétés résolubles. En déterminant des contraintes sur le twist pour générer des solutions, nous parvenons à relier des solutions connues, et nous en trouvons une nouvelle. Nous l'utilisons également pour relier des solutions avec flux de la corde hétérotique.

Nous considérons finalement des solutions de de Sitter dix-dimensionnelles. Plusieurs problèmes, dont la brisure de la SUSY, rendent la recherche de telles solutions difficile. Nous proposons un ansatz pour des sources brisant la SUSY qui aide à surmonter ces difficultés. Nous donnons alors une solution explicite sur variété résoluble, et discutons partiellement sa stabilité quatre-dimensionnelle.

Un résumé long en français de la thèse se trouve dans l'appendice D.

Contents

Chapter 1

Introduction

The standard model of particle physics is now a paradigm to describe three of the four elementary interactions of nature. This description of the electromagnetic, weak and strong interactions has been successfully tested experimentally to very high accuracy, up to energies of a few hundred GeV. Despite this situation, some aspects of this description are theoretically speaking unsatisfactory, and led in the last decades to some more advances.

One of these aspects is the question of naturalness: the standard model contains a set of parameters which are not fixed by the theory, and which can take very different values (different energy scales appear for instance). Furthermore, the model has an impressive symmetry structure whose origin is not explained. All this seems to indicate the existence of an underlying, more fundamental, structure, which would have the standard model as an effective description. One proposal in this direction have been Grand Unified Theories (GUT). From Newton and his description of gravitation, till the electroweak interaction, the idea of unification has been fruitful in theoretical physics history. This same idea is used here to unify the three interactions of the standard model into one. By extrapolating the values of their coupling constants at higher energy, one can see they almost meet at a single point around 10^{16} GeV. Therefore, one considers in the GUT models only one gauge group, which breaks at this unification scale M_U into the various gauge groups of the standard model. This is only one among many possibilities to go beyond the standard model.

Another important question about the standard model is the Higgs boson. To start with, it is the only unobserved particle in this whole picture. Accordingly, its mass is still unknown but has a lower experimental bound of 114 GeV (obtained at the LEP). To fit properly with the description of the electroweak breaking, this mass has also an upper bound of order 1 TeV. This tight window gives hopes for its discovery at the newly starting LHC. Even if its mass could get fixed experimentally, the one-loop corrections to it lead to the famous Higgs hierarchy problem. These quantum corrections turn out to be of the same order as the Higgs mass scale itself, unless an important fine-tuning is done. As we will see, this problem can be solved by supersymmetry, another possible step in the direction of an underlying structure for the standard model.

Last but not least, the gravitational interaction is not included in the standard model of particles. We know that considering gravitation is not relevant up to the Planck mass scale, $M_p \sim 10^{19}$ GeV, where this interaction becomes of the same order as the quantum effects. A fully unified description of nature including gravity is therefore not needed at the typical scales of the standard model (this claim can be relaxed in some cases like low string scale scenarios). On the contrary, it is needed to describe some extreme situations, like black holes, or early times of the universe, where microscopic quantum effects are in presence of highly curved geometries. More dramatically, such situations raise the question of quantum gravity. If gravity, viewed as a gauge theory with the Einstein-Hilbert action, is not included in the standard model, it is not only a matter of scales, but because it cannot be quantized as the other interactions are. This gauge theory is simply not renormalisable. A proper quantum theory of gravity, and more generally a unified description of all four interactions is another motivation to go beyond the standard model of particles.

Another striking discrepancy in energy scales is that of the cosmological constant with respect to the Planck mass (the so-called cosmological constant problem). Many precise cosmological measurements have led to what is now known as the standard cosmology. One of the major results of recent measurements is that our four-dimensional universe should be in accelerated expansion and be described by a de Sitter space-time, with a cosmological constant Λ of mass scale given by $M_\Lambda = \sqrt{\Lambda} \sim 10^{-12}$ GeV. That makes it more than 10^{30} order lower than the Planck mass. It is also a scale significantly lower than the standard model scales. The reason of this value and the nature of a corresponding "dark energy" are not explained, and would also require a further theoretical set-up, if not a quantum theory of gravity. As we will see in this thesis, additional space dimensions could also help in understanding the nature of this cosmological constant.

In trying to extend the standard model of particles, one often encounters supersymmetry (SUSY). This global symmetry of field theories is one of the few possible extensions of the Poincaré group (it circumvents the Coleman-Mandula no-go theorem). The corresponding (super Lie) algebra gets some fermionic generators, whose number gives the number $\mathcal{N}$ of supersymmetries: in four-dimensions, one can go from $\mathcal{N} = 1$ to $\mathcal{N} = 4$ for theories with spin one fields, and up to $\mathcal{N} = 8$ if one allows for spin two fields like the metric (the theory is then said to be maximally supersymmetric, it has 32 fermionic generators). The fields of a supersymmetric theory are arranged in multiplets having equal number of fermions and bosons, and same masses. Therefore, if such a theory was describing particle physics, each particle should have a superpartner of spin different by one-half and of same mass. It is clearly not the case at the scale of the standard model, and therefore, such a symmetry, if it exists, should be broken at some higher energy scale M_{SUSY}. Up to date, there exist many different scenarios to break SUSY.

Because of the multiplets, the only theories admitting chiral fields are $\mathcal{N} = 1$. Chirality is an important feature of the standard model so the simplest supersymmetric extensions of the standard model like the MSSM have $\mathcal{N} = 1$ SUSY. In spite of the complexity which seems to arise by allowing for SUSY (non-observed superpartners, various breaking scheme, high number of new unfixed parameters, etc.), such extensions have some important advantages. Because of this additional symmetry, the supersymmetric gauge theories are highly constrained. In particular, the quantum corrections are often much more under control, because of cancellations between bosonic and fermionic contributions. This is the reason why SUSY solves the Higgs hierarchy problem. Another consequence of these simpler quantum corrections is that the GUT models get refined: the three coupling constants meet exactly at a single point when extrapolated at higher energy. These nice features motivated the introduction of SUSY in extensions of the standard model, and the LHC will soon look for evidences of its existence at higher energy. Note that M_{SUSY} should not be much higher than the electroweak breaking scale, in order for SUSY to solve the Higgs hierarchy problem. Therefore, this scale should be in the scope of the LHC.

Another interesting feature of SUSY is to help towards the construction of a quantum gravity. When considering this symmetry local, one can construct naturally supergravity theories (SUGRA). These are supersymmetric field theories of gravity, more constrained than the standard gravity. In particular, in an eleven-dimensional space-time, the theory is unique, completely determined by the symmetries. It is a maximally supersymmetric theory, and one cannot construct any other in higher dimensions. In lower dimensions, one can construct different supergravities, which may or may not be related between them. Supergravity theories are a priori not more finite than the standard gravity, except for $\mathcal{N} = 8$ SUGRA in four dimensions. The finiteness of the latter is currently under study, some impressive cancellations appear at higher loops, in part because of SUSY, but not only. Nevertheless, the SUGRA theories play an important role in the question of quantum gravity, as they appear to be low energy effective theories of string theories.

Extensions of the standard model of particles are a priori not directly related to the problem of quantum gravity, at least from the scales point of view. As mentioned, the question of quantum gravity essentially arises at the Planck mass scale, while first extensions of the standard model are considered around the electroweak breaking scale. Nevertheless, from the unification perspective, these should be related at some point. Furthermore, if one can provide a theory of quantum gravity containing additionally some gauge groups, one could ask whether at low energy some extension of the standard model could be recovered. This is the point of view we will adopt in this thesis, the theory of quantum gravity being string theory.

String theories consider one-dimensional extended elementary objects, the strings. Their embedding in target space-time is given by their coordinates, which are taken as fields. These fields are governed by a two-dimensional conformal field theory which turns out to be a sigma-model. A string can oscillate in space-time, and these oscillation modes are quantized. A first important result of string theory has been that among the massless modes, a rank two symmetric tensor is always present. Considered as a metric, this has been a first hint of the possibility to construct a quantum theory of gravity. When including fermionic modes in the sigma-model, it turns out that the whole massless spectrum of string theory corresponds to the spectrum of some SUGRA. Furthermore, at one-loop in the conformal field theory, to avoid a conformal anomaly, one has to impose some equations which turn out to be the equations of motion of some SUGRA theory. Therefore, SUGRA corresponds to a low energy effective theory for string theory. The answer to the divergences of SUGRA are in turn given by a proper quantization in string theory. Indeed, string theory happens to be a quantum theory of gravity, even if the quantization is background dependent.

String theory has been at first defined perturbatively, specifying a particular conformal theory at each loop. When adding the fermions, anomaly cancellations at one-loop turn out to bring severe constraints on the various possible string theories. Except for some exotic scenarios, the result is that only a few string theories can be considered. Furthermore, all of them are found to be space-time supersymmetric. In that sense, SUSY is predicted by string theory: by consistency of the construction, only supersymmetric theories are allowed. The dimension of space-time is also constrained by consistency and conformal symmetry: one gets an upper bound for it. But for technical reasons, this upper bound is often preferred. This is the reason to say that superstrings live in a ten-dimensional space-time, while purely bosonic strings live in twenty-six dimensions. Out of all these constraints, only five superstring theories can be considered in ten dimensions. Two of them are $\mathcal{N}=2$ (type IIA, IIB) and the others are $\mathcal{N}=1$ (heterotic strings, or type I). All of them give a SUGRA theory (with the same name) at low energy in ten dimensions. In this thesis, we will mainly consider type II supergravities. Note that the eleven-dimensional SUGRA is then considered as the low energy effective theory of what is called M-theory, whose elementary objects cannot be strings, but rather some more extended objects like membranes, called M-branes. All these theories have been found to be related by dualities, and are now believed to be different aspects in different regimes of a single bigger theory.

String theory, first defined perturbatively, has then included non-perturbative objects called branes which can be thought of as solitons in usual field theory. They are again extended objects like membranes. In type II, a Dp-brane is geometrically a hypersurface of p dimensions evolving in time. Their discovery was much inspired by the interplay with SUGRA. At low energy, they appear as p-branes: these are particular solutions to Einstein equations corresponding to generalizations of black-hole solutions. In string theory, they turn out to be dynamical objects. Their dynamics are given perturbatively by oscillations of open strings with Dirichlet boundary conditions: the open strings end on these hypersurfaces. The effective theory of a stack of N $D3$-branes in four-dimensional Minkowski space-time is given by $\mathcal{N}=4$ super Yang-Mills theory with gauge group $SU(N)$. D-branes have led to huge advances in string theory, among which the famous AdS/CFT correspondence.

In attempts to extend the standard model, one usually adds a few ingredients at a time: SUSY, additional space dimensions, branes, etc. Various models and predictions have been constructed this way, and the LHC discoveries will hopefully help to discriminate among all these new theoretical possibilities. As discussed, we consider here a different approach: one starts from superstring theory in ten dimensions, and tries to recover the four-dimensional physics of the standard model, or some extension of it. On the way, one goes through several steps containing the additional ingredients mentioned. In this thesis, we will only focus on a few steps in this vast program. Doing so, we will also learn things from low energy theories, which may help to understand better string theory.

We mentioned that superstring theories are defined in ten-dimensional space-time. That gives six additional spacelike dimensions with respect to our observed four-dimensional space-time. In the most common scenario, these additional dimensions are said to be compact, meaning that they are not extended as the observed ones are, but closed on themselves. These six dimensions are then assumed to form a smooth compact manifold M. The simplest example is probably a product of six circles, or equivalently a six-dimensional torus T^6. This six dimensions being unobserved, their typical scale (the mean radius for instance) should be beyond the scope of our past experiments. Roughly speaking, M is too small to be detected; this is the reason why we call it the internal space. Its size gives an additional scale, the compactification scale M_c. Note that one could a priori consider several different scales according to these different additional dimensions. For simplicity here we only consider one. The ten-dimensional theory then has to go through a procedure known as the dimensional reduction, or compactification, in order to provide a four-dimensional theory. Before describing this procedure in more details, let us discuss further the various energy scales appearing.

Starting from string theory, we first consider ten-dimensional SUGRA as its low energy description. For this to be valid, the massive modes of string theory have to be too heavy to contribute. Their mass scale is typically given by the string length l_s. Note that this parameter, the only one of string theory, is unfixed. It is often taken to be the same as the Planck scale, but there is a priori no reason for this choice. For the SUGRA approximation to be valid, the coupling constant of string theory g_s also needs to be small (for the description to stay perturbative).

Considering SUGRA in ten dimensions and lowering further the energy, one then has to dimensionally reduce it as discussed. As discussed previously, obtaining in four dimensions an $\mathcal{N} = 1$ supersymmetric theory could be a way to match with some extensions of the standard model. These extensions have their own complexity, but they also present some advantages we mentioned. Therefore, typical compactification schemes try to preserve some supersymmetry. This requirement gives important constraints on the choice of the internal manifold M, as we will see in more details. In terms of scales, it also means that the breaking of SUSY is chosen to occur at an energy lower than the compactification scale.

To summarize, we have the following general hierarchy in energy scales, and accordingly the proposed effective theories:

SM + Grav.	M_{SUSY}	MSSM + Grav. ? $4d$ SUGRA + ... ?	M_c	$10d$ SUGRA	$M_s = \frac{1}{l_s} \stackrel{?}{=} M_p$	superstring theory	E

Note that the unification scale M_U is not present in this graph, even if it should be found between the SUSY breaking scale M_{SUSY} and the string scale M_s. Its absence is related to the undefined description between those scales, or more precisely between M_{SUSY} and M_c. The effective theory in this range really depends on the model under consideration, and this is waiting for experimental confrontation. Let us discuss particular cases of importance for us. Heterotic string contains non-abelian gauge groups. These are big enough to be used in a GUT scenario and reproduce at the end of the day the gauge interactions of the standard model, even if this program has not yet been completed. On the contrary, type II SUGRA only contains abelian gauge fields in its (perturbative) spectrum. As mentioned, the way to obtain non-abelian gauge groups is to add stacks of D-branes. Therefore, there is in type II a whole expertise about how to add branes properly in order to construct a model which will have the desired content in fields, gauge groups, couplings, etc. Nowadays, type

II is considered more promising than the heterotic approach, even if it was not the case before the AdS/CFT correspondence. Let us also mention the more recent approach of F-theory (related in some extent to type II) which seems to lead to even more realistic models, by obtaining interesting values for the Yukawa couplings.

In this thesis, we will remain a step before this model building. When one is interested in low energy effective theories, one should first find what are the light (or massless) modes of the full theory. These are often determined *around a background*, meaning one considers the small (light) fluctuations of the theory around a background, or equivalently around the vacuum of a potential. Then, these modes only are considered, i.e. the theory is truncated to these modes, when this can be done consistently. The theory obtained is a low energy effective theory. This is also the idea behind the model building, where the construction is done over a given background, and the modes added are supposed to be light. In this thesis, we will restrict ourselves to finding appropriate backgrounds of type II ten-dimensional SUGRA. This means getting interesting solutions of the ten-dimensional equations of motion. Further steps to be done would be to reconstruct the effective (four-dimensional) theory over it, if not a whole model for phenomenology.

Let us now give more details about the dimensional reduction, where light modes play a crucial role. The idea of additional dimensions and the dimensional reduction goes back the twenties, with the work done by Kaluza and Klein. Let us consider a field theory on a four-dimensional space-time times a fifth spacelike dimension given by a circle. One can expand the fields in a Fourier serie in this fifth direction, the different terms in the expansion corresponding to quantized momenta along the circle. These momenta are of the form $\frac{n}{R}$, where n is the integer of the serie, and R is the radius of the circle. Suppose the fields expanded are massless in five dimensions. Then, each momentum gives the four-dimensional mass for each of the modes of the expansion. One can then integrate the action over the fifth direction, to get a four-dimensional action. The more the mass (the n) is important, the less the term will contribute. Furthermore, the smaller R is, the less the $n \neq 0$ terms will contribute. Therefore, if R is small compared to the four-dimensional scales, or if the mass $\frac{1}{R}$ is big with respect to the other involved in the theory, one can neglect the non-zero modes. The theory is truncated to the four-dimensional massless modes. This is called the Kaluza-Klein reduction: in the five-dimensional action, one simply does not consider the dependence of the fields on the internal dimensions, which is equivalent to considering the internal length scales to be small, or also keeping only the four-dimensional massless modes. Note that in some cases, one has to be more careful: first determine what are the light modes (which are not always massless), and then check whether it is possible to get rid of interactions with massive modes, before truncating the spectrum. We will not work out in details this dimensional reduction, but only determine interesting backgrounds, on which one should further perform it. Nevertheless, note that out of this general procedure, one often gets scalar fields in four dimensions. Indeed, each internal mode which has no four-dimensional vector or spinor index simply gives a scalar after the integration over internal dimensions. Such massless scalar fields are called moduli, we will come back to them.

We explained so far the general scheme of the approach considered. Motivated by the nice features of string theory (in particular its answer to the quantum gravity question), one should first consider the low energy effective theory given by SUGRA. Then one finds a ten-dimensional interesting background, determines the light modes over it, truncates the theory to them, and then dimensionally reduces towards four-dimensions. One should finally extend this reduction by constructing a phenomenologically viable model, giving a possibly supersymmetric extension of the standard model. In this thesis, we focus only on one step of this program, which is to get a interesting background. To do so, the choice of the internal manifold M is of particular importance. Let us now enter deeper in the formalism, and give more details on this question.

Chapter 2

Supersymmetric vacua and Generalized Complex Geometry

The main part of this thesis is devoted to the study of supersymmetric solutions of type II supergravity with some non-trivial fluxes, corresponding to compactifications of ten-dimensional supergravity to four dimensions. The conditions for having a supersymmetric vacuum constrains the geometry of the internal manifold M. For the well-known case of fluxless compactifications, minimal supersymmetry requires the internal manifold to be a Calabi-Yau. For flux backgrounds, the supersymmetry conditions can be cast in a simple form [4, 5] using the formalism of Generalized Complex Geometry (GCG), recently developed by Hitchin and Gualtieri [6, 7]: the internal manifold is then characterized to be a Generalized Calabi-Yau (GCY).

In this chapter we briefly introduce the main ingredients we will need in the rest of the thesis to explicitly look for flux vacua. We first discuss ten-dimensional type II supergravities (SUGRA). Then we give the general ansatz for the solutions describing compactifications to four dimensions. Basics of Generalized Complex Geometry are then introduced and in particular the notion of pure spinors. Then we present the supersymmetry conditions for flux vacua rewritten in terms of pure spinors. We end the chapter with an outline of the thesis. Conventions, and more details are given in appendix A.

2.1 Type II supergravities

In this thesis, we will mainly consider type IIA and type IIB supergravities. These are the ten-dimensional effective theories for the massless fields of type II string theories. The bosonic sector consists of the Neveu-Schwarz Neveu-Schwarz (NSNS) and Ramond-Ramond (RR) fields. The NSNS sector is the same for both theories: it contains the metric, the dilaton ϕ and the NSNS two-form B. The latter is a $U(1)$ gauge potential with field strength $H = \mathrm{d}B$. The RR sector depends on the theory. It contains odd forms in IIA, and even forms in IIB. These are again $U(1)$ gauge potentials. We will use the democratic formulation [8], which considers all RR potentials C_n with $n = 1, \ldots, 9$ for IIA and C_n with $n = 0, \ldots, 8$ for IIB. These potentials are not all independent: for instance C_2 is equivalent to C_6 because of Hodge duality. Therefore to reduce to the independent degrees of freedom we will impose a self-duality constraint on the field strengths

$$F_n = (-1)^{\mathrm{int}[n/2]} \; \hat{*} F_{10-n} \, . \tag{2.1.1}$$

where $\hat{*}$ is the ten-dimensional Hodge star and with

$$\mathrm{IIA} \quad : \quad F_2 = \mathrm{d}C_1 + B \; F_0 \; , \; F_4 = \mathrm{d}C_3 - H \wedge C_1 + \frac{1}{2} B \wedge B \; F_0 \; , \tag{2.1.2}$$

$$\mathrm{IIB} \quad : \quad F_1 = \mathrm{d}C_0 \; , \; F_3 = \mathrm{d}C_2 - H \; C_0 \; , \; F_5 = \mathrm{d}C_4 - H \wedge C_2 \; . \tag{2.1.3}$$

In a more compact notation

$$F = (\mathrm{d} - H\wedge)C + e^{B} F_0 \ , \tag{2.1.4}$$

where F and C are the sums of RR field strengths and potentials in each theory, and the F_0 is the Roman mass term that can be added in the IIA theory. The exponential has to be understood as developed with the wedge product

$$e^{B} = 1 + B + \frac{1}{2} B \wedge B + \dots \ . \tag{2.1.5}$$

The fermionic sector consists of two Majorana-Weyl spin 3/2 spinors, the gravitinos ψ^i_M, and two Majorana-Weyl spin 1/2 spinors, the dilatinos $\tilde{\lambda}^i$. Gravitinos and dilatinos have opposite chirality. In IIA the gravitinos (hence the dilatinos) have opposite chirality, while in two IIB the gravitinos have the same chirality, which we choose positive. Correspondingly the dilatinos will have negative chirality.

Both IIA and IIB theories are maximally supersymmetric, meaning they have 32 supercharges and so have $\mathcal{N}=2$ supersymmetries. They differ in their chirality: type IIA is non-chiral while type IIB is chiral.

Let us give here simply the bosonic sector of the type IIA action, which we will need in chapter 5. We mostly follow the conventions of [9, 10]; we differ in the definition of the Hodge star (see convention in appendix A.1) where we have an extra sign depending on the parity of the forms[1]. The bosonic action of type IIA in string frame is given by

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^{10}x \sqrt{|g_{10}|}\ [e^{-2\phi}(R_{10} + 4|\nabla\phi|^2 - \frac{1}{2}|H|^2) - \frac{1}{2}(|F_0|^2 + |F_2|^2 + |F_4|^2)] \ , \tag{2.1.6}$$

where $2\kappa^2 = (2\pi)^7 (\alpha')^4$, $\alpha' = l_s^2$, and

$$F_k \wedge \hat{*} F_k = \mathrm{d}^{10}x\ \sqrt{|g_{10}|}(-1)^{(10-k)k}\ \frac{F_{\mu_1\dots\mu_k}F^{\mu_1\dots\mu_k}}{k!} = \mathrm{d}^{10}x\ \sqrt{|g_{10}|}(-1)^{(10-k)k}\ |F_k|^2 \ . \tag{2.1.7}$$

$|g_{10}|$ denotes the determinant of the ten-dimensional metric and $\hat{*}$ the ten-dimensional Hodge star, reserving the symbol $*$ for its six-dimensional counterpart.

To this action we could add a topological part called the Chern-Simons term, but also possible source actions S_s. For a discussion of these terms, we refer to chapter 5.

Let us now give the equations of motion (e.o.m.) of type IIA supergravity in string frame. The ten-dimensional Einstein and dilaton equations are given by

$$\begin{aligned} R_{MN} - \frac{g_{MN}}{2} R_{10} = & -4\nabla_M\phi\nabla_N\phi + \frac{1}{4} H_{MPQ} H_N{}^{PQ} + \frac{e^{2\phi}}{2}\left(F_{2\ MP} F_{2\ N}{}^{P} + \frac{1}{3!} F_{4\ MPQR} F_{4\ N}{}^{PQR}\right) \\ & - \frac{g_{MN}}{2}\left(-4|\nabla\phi|^2 + \frac{1}{2}|H|^2 + \frac{e^{2\phi}}{2}(|F_0|^2 + |F_2|^2 + |F_4|^2)\right) + e^{\phi}\ \frac{1}{2} T_{MN} \ , \end{aligned} \tag{2.1.8}$$

$$8(\nabla^2\phi - |\nabla\phi|^2) + 2R_{10} - |H|^2 = -e^{\phi}\frac{T_0}{p+1} \ , \tag{2.1.9}$$

where M, N, P, Q, R are ten-dimensional space-time indices. T_{MN} and T_0 are the source energy momentum tensor and its partial trace, respectively[2], with $p+1$ being the worldvolume dimension

[1] In IIA, the sign is always positive on RR fields, but not on the odd forms, H and $\mathrm{d}\phi$, hence the sign difference with respect to [10] for the corresponding terms in the action, when expressed with the Hodge star. The sign difference is related to the fact we use the Mukai pairing, defined in (2.4.14), to give the norm (see below equation (2.4.13) and appendix B.3): for a real form α, we have $\langle *\lambda(\alpha), \alpha\rangle = |\alpha|^2 \times \mathrm{vol}$, where λ is defined in (2.2.3), and vol is the volume form.

[2] In our conventions

$$\frac{1}{\sqrt{|g_{10}|}}\frac{\delta S_s}{\delta\phi} = -\frac{e^{-\phi}}{2\kappa^2}\frac{T_0}{p+1} \ , \qquad \frac{1}{\sqrt{|g_{10}|}}\frac{\delta S_s}{\delta g^{MN}} = -\frac{e^{-\phi}}{4\kappa^2} T_{MN} \ . \tag{2.1.10}$$

of the source. The e.o.m. of the fluxes are given by

$$\begin{aligned}
&\mathrm{d}(e^{-2\phi}\hat{*}H) + F_0 \wedge \hat{*}F_2 + F_2 \wedge \hat{*}F_4 + \frac{1}{2}F_4 \wedge F_4 = \text{source term} \\
&(\mathrm{d} + H\wedge)(\hat{*}F) = 0 \ ,
\end{aligned} \tag{2.1.11}$$

We will come back in chapter 5 to the source term in the H e.o.m. (see [11]).

The fluxes of type II supergravities are also subject to the Bianchi identities (BI), given by (we will not consider any $NS5$-brane)

$$\begin{aligned}
&\mathrm{d}H = 0 \\
&(\mathrm{d} - H\wedge)F = \delta(\text{source}) \ ,
\end{aligned} \tag{2.1.12}$$

where $\delta(\text{source})$ is the charge density of RR sources: these are D-branes or orientifold planes (O-planes).

2.2 The internal manifold

In this thesis we are interested in compactifications to four dimensions, where the four dimensional space in maximally symmetric: Minkowski, Anti de Sitter or de Sitter spaces. To this extent, we can make some hypothesis on the form of the solutions we are looking for.

2.2.1 Supersymmetric vacuum on four plus six dimensions

Splitting in four plus six

We will consider the ten-dimensional space-time to be a warped product of a maximally symmetric four-dimensional space-time and a six-dimensional compact space M. Then, for the ten-dimensional metric, we take

$$ds^2_{(10)} = e^{2A(y)} \; g_{\mu\nu}dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n \ , \tag{2.2.1}$$

where e^{2A} is the warp factor depending on the internal dimensions y^m. The four-dimensional metric of signature $(-1,+1,+1,+1)$ will have Poincaré, $SO(1,4)$ or $SO(2,3)$ symmetry for M_4, AdS_4 or dS_4, respectively.

For the RR and NSNS fluxes, we can a priori allow them to have non-zero background values. Nevertheless, maximal symmetry in four-dimensions requires the fluxes be non-trivial only on the internal manifold

$$F_k^{(10)} = F_k + \mathrm{vol}_4 \wedge \lambda(*F_{6-k}) \, . \tag{2.2.2}$$

Here vol_4 is the warped four-dimensional volume form and λ acts on any p-form A_p by a complete reversal of its indices

$$\lambda(A_p) = (-1)^{\frac{p(p-1)}{2}} A_p \ . \tag{2.2.3}$$

We define the total internal RR field F as

$$\mathrm{IIA} \quad : \quad F = F_0 + F_2 + F_4 + F_6 \ , \tag{2.2.4}$$

$$\mathrm{IIB} \quad : \quad F = F_1 + F_3 + F_5 \ , \tag{2.2.5}$$

with F_k the internal k-form RR field strength. With this ansatz the flux equations of motion and Bianchi identities become

$$\begin{aligned}
&\mathrm{d}(e^{4A-2\phi} * H) \pm e^{4A} \sum_k F_k \wedge *F_{k+2} = \text{source term} \ , \qquad && \mathrm{d}H = 0 \ , \\
&(\mathrm{d} + H\wedge)(e^{4A} * F) = 0 \ , && (\mathrm{d} - H\wedge)F = \delta_s \ ,
\end{aligned} \tag{2.2.6}$$

where the upper/lower sign is for IIA/B, and now d and $*$ are internal. δ_s denotes the contribution from sources. Flux compactifications to four-dimensional Minkowski or de Sitter are not possible without sources with negative tensions, O-planes. These are needed to cancel the positive flux contribution to the trace of the energy-momentum tensor [12, 13]. In particular, because of maximal symmetry in four dimensions we will consider space-filling D-branes or O-planes.

Finally, imposing maximal symmetry also sets all the vacuum expectation values of the fermionic fields to zero. So we will look for purely bosonic solutions.

Supersymmetric solutions

We will actually restrict even more the form of our solutions by imposing that they have minimal supersymmetry, namely $\mathcal{N}=1$ in four dimensions.

From a physical point of view this choice corresponds to the hypothesis that supersymmetry is spontaneously broken at low energies (see chapter 1). From a technical point of view, looking for supersymmetric solutions simplifies the resolution. Indeed, for Minkowski backgrounds, it has been proven in [14, 15, 11] that all equations of motion are implied once the Bianchi identities and the ten-dimensional SUSY conditions are satisfied. So, instead of solving the equations of motion, which are second order differential equations, one can solve a set of first order equations.

For a purely bosonic background, the conditions for unbroken supersymmetry is that the variations of the fermionic fields vanish. Then, in type II supergravity we have to set to zero the bosonic part of the gravitino and dilatino SUSY variations

$$\delta\psi_M = 0 \, , \qquad\qquad \delta\tilde{\lambda} = 0 \, . \tag{2.2.7}$$

We work in string frame and use the democratic formulation. If we write the two gravitino and the two dilatino as doublets $\psi_M = (\psi^1_M, \psi^2_M)$ and $\tilde{\lambda} = (\tilde{\lambda}^1, \tilde{\lambda}^2)$, then their supersymmetry variations read

$$\delta\psi_M = (D_M + \frac{1}{4} H_M \mathcal{P})\epsilon + \frac{1}{16} e^{\phi} \sum_n \not{F}^{(2n)}\, \Gamma_M \mathcal{P}_n \, \epsilon \, , \tag{2.2.8}$$

$$\delta\tilde{\lambda} = (\not{\partial}\phi + \frac{1}{2} \not{H} \mathcal{P})\epsilon + \frac{1}{8} e^{\phi} \sum_n (-1)^{2n} (5-2n) \not{F}^{(2n)}\, \mathcal{P}_n \epsilon \, , \tag{2.2.9}$$

where the supersymmetry parameter $\epsilon = (\epsilon^1, \epsilon^2)$ is also a doublet of Majorana-Weyl spinors. The matrices $\mathcal{P}$ and $\mathcal{P}_n$ are different in IIA and IIB. For IIA $\mathcal{P} = \Gamma_{11}$ and $\mathcal{P}_n = \Gamma_{11}\sigma_1$, while in IIB $\mathcal{P} = -\sigma_3$, $\mathcal{P}_n = \sigma_1$ for $n + 1/2$ even and $i\sigma_2$ for $n + 1/2$ odd.

In our ansatz for the metric and fluxes we took the space-time to be the product of four plus six dimensions. Then the ten-dimensional SUSY parameters ϵ^1 and ϵ^2 have to be decomposed accordingly. With the metric (2.2.1), the Lorentz group is broken to $SO(1,3) \times SO(6)$ and the parameters $\epsilon^{i=1,2}$ are decomposed as

$$\begin{aligned} \epsilon^1 &= \zeta^1 \otimes \sum_a \alpha^1_a \eta^1_a + c.c. \, , \\ \epsilon^2 &= \zeta^2 \otimes \sum_a \alpha^2_a \eta^2_a + c.c. \, . \end{aligned} \tag{2.2.10}$$

ζ^i are the four-dimensional chiral spinors corresponding to the SUSY parameters in four dimensions. Similarly η^i_a are $SO(6)$ Weyl spinors. The six-dimensional (internal) spinors can be seen, from the four-dimensional point of view, as internal degrees of freedom of the ζ^i. So the number $\mathcal{N}$ of four-dimensional SUSYs is given by the number of components of the internal spinors satisfying the SUSY equations. In six dimensions, there can be at most (on a parallelizable manifold) four solutions η^1_a for each chirality. So the maximum $\mathcal{N}$ is clearly $\mathcal{N}=8$. To get an $\mathcal{N}=1$ vacuum as we want, one needs a pair (η^1, η^2) of internal spinors that satisfy the SUSY conditions (and for $\mathcal{N}=1$ one also needs

$\zeta^1 = \zeta^2$). Given the chiralities of the two theories, let us then consider the following decomposition in IIA

$$\begin{aligned} \epsilon^1 &= \zeta^1_+ \otimes \eta^1_+ + \zeta^1_- \otimes \eta^1_- \ , \\ \epsilon^2 &= \zeta^2_+ \otimes \eta^2_- + \zeta^2_- \otimes \eta^2_+ \ , \end{aligned} \tag{2.2.11}$$

and this one in IIB

$$\epsilon^{i=1,2} = \zeta^i_+ \otimes \eta^i_+ + \zeta^i_- \otimes \eta^i_- \ , \tag{2.2.12}$$

where the complex conjugation changes the chirality: $(\eta_+)^* = \eta_-$.

Typically one further requires the internal spinors to be globally defined. A justification for this assumption comes from the computation of the four-dimensional effective actions obtained by reducing the ten-dimensional theory on the internal manifold. In order to have supersymmetry in four dimensions, we need a globally defined basis of internal spinors on which to reduce the SUSY parameters. This topological requirement can be translated in terms of G-structures that we will now discuss.

2.2.2 Internal spinors and G-structures

For a manifold M, the structure group is the group in which take values the structure functions. In six dimensions, the structure group is a priori $GL(6)$. The existence of globally defined tensors or spinors leads to a reduction of the structure group. The manifold is said to admit a G-structure when its structure group is reduced to the subgroup G. For instance, given a metric and an orientation, it is reduced to $SO(6) \sim SU(4)$. In presence of one globally defined spinor, it is further reduced to $SU(3)$, and to $SU(2)$ if a second (independent) globally defined spinor exists. In this thesis we will often consider $SU(3)$ and $SU(2)$ structures.

In six dimensions, an $SU(3)$ structure is defined by a globally defined Weyl spinor η_+. Here we take η_+ of positive chirality and of unitary norm. The complex conjugation changes the chirality: $(\eta_+)^* = \eta_-$. A G-structure is equivalently defined in terms of G-invariant globally defined forms. These can be obtained as bilinears of the globally defined spinors. For an $SU(3)$ structure, one can define a holomorphic three-form Ω and a Kähler form J given by[3]

$$\begin{aligned} \Omega_{\mu\nu\rho} &= -i\eta_-^\dagger \gamma_{\mu\nu\rho}\eta_+ \ , \\ J_{\mu\nu} &= -i\eta_+^\dagger \gamma_{\mu\nu}\eta_+ \ , \end{aligned} \tag{2.2.13}$$

satisfying the structure conditions

$$J \wedge \Omega = 0 \qquad \frac{4}{3}J^3 = i\Omega \wedge \overline{\Omega} \neq 0 \ . \tag{2.2.14}$$

Similarly, an $SU(2)$ structure is defined by two orthogonal globally defined spinors η_+ and χ_+ (we take them of unitary norm). In terms of invariant forms, an $SU(2)$ structure is given by a holomorphic one-form[4] z (we take $||z||^2 = 2$), a real two-form j and a holomorphic two-form ω

$$\begin{aligned} z_\mu &= \eta_-^\dagger \gamma_\mu \chi_+ \ , \\ j_{\mu\nu} &= -i\eta_+^\dagger \gamma_{\mu\nu}\eta_+ + i\chi_+^\dagger \gamma_{\mu\nu}\chi_+ \ , \\ \omega_{\mu\nu} &= \eta_-^\dagger \gamma_{\mu\nu}\chi_- \ , \end{aligned} \tag{2.2.15}$$

satisfying the following structure conditions

$$j^2 = \frac{1}{2}\, \omega \wedge \overline{\omega} \neq 0 \ , \tag{2.2.16}$$

$$j \wedge \omega = 0 \ , \qquad \omega \wedge \omega = 0 \ , \tag{2.2.17}$$

$$z \llcorner \omega = 0 \ , \qquad z \llcorner j = 0 \ . \tag{2.2.18}$$

[3]The indices $\mu,\ \nu,\ \rho$ are real.

[4]Note that it is possible to rewrite the spinor χ_+ as $\chi_+ = \frac{1}{2} z\eta_-$.

The definition of the contraction $\llcorner$ is given in appendix A.1. We give one possible derivation of these structure conditions in appendix A.2.1.

Note that the $SU(2)$ structure is naturally embedded in the $SU(3)$ structure defined by η_+:

$$J = j + \frac{i}{2} z \wedge \overline{z} \, , \qquad \Omega = z \wedge \omega \quad \Rightarrow \quad j = J - \frac{i}{2} z \wedge \overline{z} \, , \qquad \omega = \frac{1}{2} \overline{z} \llcorner \Omega \, . \tag{2.2.19}$$

The topological requirement given by the existence of the two globally defined internal spinors $\eta_+^{i=1,2}$ is then equivalent to the existence of globally defined forms, satisfying the structure conditions. Similarly, the relation between globally defined spinors and globally defined forms provides a useful alternative way to express the differential requirement given by the SUSY conditions in terms of the structure forms. As an example we will first discuss the case of fluxless compactifications, where supersymmetry leads to the Calabi-Yau condition.

2.2.3 Calabi-Yau manifold

Let us consider solutions where no flux is present: $F_k = 0$, $H = 0$. Moreover we assume that the manifold only admits a single globally defined spinor $\eta_+^1 = \eta_+^2 = \eta_+$. Then one can a priori get an $\mathcal{N} = 2$ theory in four dimensions with $\zeta^{1,2}$. With these hypothesis, by decomposing the spinors and the SUSY conditions (2.2.8) and (2.2.9) into four and six-dimensional parts, one gets

$$\partial_\mu \zeta_+ = 0 \ , \ D_m \eta_+^1 = 0 \ , \tag{2.2.20}$$

where m is an internal index. This means the internal manifold requires not only the existence of a globally defined spinor, but the latter needs to be covariantly constant. This means that the holonomy group of M is reduced to $SU(3)$ and therefore M must be a CY three-fold CY_3 [16].

In terms of the G-structures, we get an $SU(3)$ structure. In addition, the closure of the spinor translates into differential conditions for the forms defining the $SU(3)$ structure. The structure forms are closed:

$$\mathrm{d}J = 0 \ , \ \mathrm{d}\Omega = 0 \ . \tag{2.2.21}$$

These conditions give the integrability of both the almost complex and symplectic structures, meaning the internal manifold needs to be Kähler. This is indeed the case of a CY. Another property of a Calabi-Yau manifold is that it admits a Ricci-flat metric[5] $R_{mn} = 0$.

In absence of fluxes, looking for a SUSY vacuum requires to consider a CY_3 manifold as the internal manifold. By reducing the type II actions on the Calabi-Yau will give an $\mathcal{N} = 2$ effective theory in four dimensions (a similar reduction for the heterotic string leads to an $\mathcal{N} = 1$ effective four-dimensional theory). We will not dwell into the details of the four-dimensional effective actions. There are two aspects we would like to stress. The first is the fact that the geometry of the internal manifold is crucial to determine the field content and symmetries of four-dimensional theory. The second is that the effective theories constructed by reducing on a Calabi-Yau manifold suffer of the presence of massless fields which are not constrained by any potential. These are the so-called moduli. In supersymmetric theories massless scalar fields are not a problem, the trouble is if some of them stay massless after SUSY breaking: massless scalar fields would provide long range interactions which should be observed, except for a confinement scenario.

[5]This can be seen from the integrability condition on the internal spinor

$$0 = [D_m, D_n]\eta_+ = \frac{1}{4} R_{mn}{}^{pq} \gamma_{pq} \eta_+ \qquad \Rightarrow \qquad \gamma^n \gamma^{pq} R_{mnpq} \eta_+ = 0 \, . \tag{2.2.22}$$

Using the gamma matrix identity $\gamma^i \gamma^{jk} = \gamma^{ijk} + g^{ij}\gamma^k - g^{ik}\gamma^j$ and $R_{ijkl} + R_{iklj} + R_{iljk} = 0$, we have

$$0 = \gamma^n \gamma^{pq} R_{mnpq} \eta_+ = \gamma^{npq} R_{mnpq} \eta - 2 R_{mn} \gamma^n \eta \, . \tag{2.2.23}$$

This equation reduces $R_{mn} = 0$, since the first term is zero because of the symmetries of the curvature tensor.

The idea is then to look for mechanisms allowing to stabilise at least some of the moduli already in the supersymmetric theory. Suppose that scalar fields φ appearing in the effective theory have a potential $V(\varphi)$. If it admits a minimum for some value φ_0, then the action and the potential can be developed around it as

$$V(\varphi) \approx V(\varphi_0) + V''(\varphi_0)(\partial\varphi)^2 \ . \tag{2.2.24}$$

Therefore, giving a vacuum value to the scalar (fixing it) gives a mass (term) to it. Provided this mass is sufficiently high, one could integrate the scalar fields out, and so get rid of them. As already mentioned, CY compactifications do not generate a potential for the scalars.

2.2.4 Flux vacua

The moduli problem led in years 2000' to the development of flux compactifications: finding solutions in presence of non-trivial values for the background fluxes. Background fluxes on the internal manifold are interesting because they are known to generate a potential which can fix some, if not all (in some AdS compactifications) the moduli. The remaining moduli are most of the time fixed by non-perturbative contributions. See [17] for reviews on the subject.

The presence of fluxes drastically changes the properties of the solutions. Basically, fluxes on the internal manifold back react via their energy density, and therefore, M can a priori not be flat anymore. In particular, the internal space is no longer Calabi-Yau: $R_{mn} \neq 0$. The presence of fluxes also modifies the SUSY conditions, as we can see from (2.2.8) and (2.2.9). When going to the internal space, one typically gets non-zero right handsides in (2.2.20) or in (2.2.21) that depend on the fluxes. For instance the internal components of the gravitino become

$$\delta\psi_m^1 = (D_m + \frac{1}{4}H_m)\eta_+^1 + \not{F}_m\eta_+^1 + \not{F}_m\eta_+^2 \ , \tag{2.2.25}$$

$$\delta\psi_m^2 = (D_m + \frac{1}{4}H_m)\eta_+^2 + \not{F}_m\eta_+^2 + \not{F}_m\eta_+^1 \ . \tag{2.2.26}$$

One of the oldest example of flux vacua is given by heterotic backgrounds with non-zero H-flux [18, 19]. Already in this context one can see that, in presence of fluxes, the manifold turned out to be only complex since J is not closed anymore (see (2.2.21)). For certain flux solutions, the manifold does not differ much from a CY: it can be only a conformal CY, where the conformal factor can be taken in some limit to 1. But in other cases like the heterotic example, the deviation from the CY can be more dramatic. The topology can differ, which makes no smooth limit to the CY possible. The typical example is the twisted tori: a non-trivial fibration of circles over a torus. We will study some of them (nilmanifolds, solvmanifolds) in more details.

It is therefore natural to ask if, for flux backgrounds, it is still possible to say something about the geometry of the internal manifold. A mathematical characterisation of the internal manifold has been given for type II compactifications to Minkowski [4, 5]: in presence of fluxes, the internal manifold has to be a Generalized Calabi-Yau (GCY). This definition relies on the formalism of Generalized Complex Geometry (GCG), a mathematical framework recently developed by Hitchin and Gualtieri [6, 7]. The Generalized Calabi-Yau condition is obtained by a rewriting of the SUSY conditions in presence of fluxes in terms GCG objects. This condition on M is unfortunately only necessary, on the contrary to the CY condition. As we will see later on, the remaining conditions are mainly due to the presence of RR fields, which are not really incorporated in this formalism. Note that further developments to include them have been proposed [20, 21, 22].

GCG can describe manifolds which are complex, symplectic, or partially complex and partially symplectic. One can define a more general structure, the generalized complex structure, which incorporates all the previous cases in one formalism and this helps to understand the zoology of manifolds appearing in presence of fluxes. For instance, nilmanifolds, even if they do not always have a complex or symplectic structure, are proved to be all GCY (which is a subcase of generalized complex) [23]. So they will be of particular interest for us.

As we will see in more details, this formalism has the advantage to incorporate in a natural way an $O(6,6)$ action which includes the T-duality group. We would like to mention that Generalized Complex Geometry has interesting applications, not only in supergravity, but also in the world-sheet approach to string theory. For these applications see for instance [24, 25, 26].

2.3 Basics of Generalized Complex Geometry

In this section, we give basic notions of GCG that will be needed in the rest of the thesis. We will particularly discuss the $O(d,d)$ action, generalized vielbein and pure spinors. The latter will then be used to rewrite the SUSY conditions and show that a necessary condition for $\mathcal{N}=1$ SUSY vacua is for the internal manifold to be a Generalized Calabi-Yau.

2.3.1 Generalized tangent bundle and $O(d,d)$ transformations

Generalized complex geometry is the generalization of complex geometry to the sum of the tangent and cotangent bundle of a manifold

$$TM \oplus T^*M \tag{2.3.1}$$

Thus it treats vectors and one-forms on the same footing. Sections of $TM \oplus T^*M$ are called generalized vectors and we will denote them as

$$V = (v + \xi) \in TM \oplus T^*M \tag{2.3.2}$$

In fact it is possible to include a two-form potential B by defining a connective structure of a gerbe. Then one considers the generalized tangent bundle E of a d-dimensional manifold M, where E is a non-trivial fibration of T^*M over TM

$$\begin{array}{ccc} T^*M & \hookrightarrow & E \\ & & \downarrow \\ & & TM \end{array} \tag{2.3.3}$$

The sections of E are called generalized vectors and can be written locally as the sum of a vector and a one-form

$$V = v + \xi = \begin{pmatrix} v \\ \xi \end{pmatrix} \in TM \oplus T^*M\,. \tag{2.3.4}$$

The patching of the generalized vectors between two coordinate patches U_α and U_β is given by

$$v_{(\alpha)} + \xi_{(\alpha)} = A_{(\alpha\beta)} v_{(\beta)} + \left[A^{-T}_{(\alpha\beta)} \xi_{(\beta)} + \iota_{A_{(\alpha\beta)} v_{(\beta)}} \mathrm{d}\Lambda_{(\alpha\beta)} \right]\,. \tag{2.3.5}$$

or in vector notation

$$\begin{pmatrix} v \\ \xi \end{pmatrix}_{(\alpha)} = \begin{pmatrix} A & 0 \\ -\mathrm{d}\Lambda\ A & A^{-T} \end{pmatrix}_{(\alpha\beta)} \begin{pmatrix} v \\ \xi \end{pmatrix}_{(\beta)}\,. \tag{2.3.6}$$

$A_{(\alpha\beta)}$ is an element of $GL(d,\mathbb{R})$, and gives the usual patching of vectors and one-forms. To simplify notations we set $A^{-T} = (A^{-1})^T$. We denote by ι a contraction. $\mathrm{d}\Lambda_{(\alpha\beta)}$ is a two-form which gives an additional shift to the one-form. It is due to the non-trivial fibration of T^*M over TM. The one-form $\Lambda_{(\alpha\beta)}$ satisfies

$$\Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = g_{(\alpha\beta\gamma)} \mathrm{d} g_{(\alpha\beta\gamma)} \tag{2.3.7}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$ and $g_{(\alpha\beta\gamma)}$ is a $U(1)$ element. Therefore, the shift $\mathrm{d}\Lambda_{(\alpha\beta)}$ corresponds to a gauge transformation of the B-field, when going from U_α to U_β.

From the string theory point view, it is very natural to consider such a connective structure: the two-form B will correspond to the B-field of SUGRA.

E is equipped with a natural metric, defined by the coupling of vectors and one-forms

$$\eta(V,V) = \iota_v \xi \qquad \Leftrightarrow \qquad V^T \eta V = \frac{1}{2} \begin{pmatrix} v & \xi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ \xi \end{pmatrix} . \tag{2.3.8}$$

The metric is left invariant by $O(d,d)$ transformations

$$O^T \eta O = \eta . \tag{2.3.9}$$

where O is a $2d \times 2d$ matrix

$$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.3.10}$$

satisfying

$$a^T c + c^T a = 0 \tag{2.3.11}$$
$$b^T d + d^T b = 0 \tag{2.3.12}$$
$$a^T d + c^T b = \mathbb{I} . \tag{2.3.13}$$

$O(d,d)$ transformations act on the generalized vectors in the fundamental representation

$$V' = OV = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v \\ \xi \end{pmatrix} . \tag{2.3.14}$$

The $O(d,d)$ group can generated by $GL(d)$ transformations, B-transforms, and β-transforms, respectively given by

$$\begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} , \begin{pmatrix} \mathbb{I} & 0 \\ B & \mathbb{I} \end{pmatrix} , \begin{pmatrix} \mathbb{I} & \beta \\ 0 & \mathbb{I} \end{pmatrix} , \tag{2.3.15}$$

where $A \in GL(d)$, and B and β are $d \times d$ antisymmetric matrices. When acting on forms, the last two elements correspond respectively to a two-form and a bi-vector (acting with two contractions), as we will see with the spinorial representation.

A famous example of $O(d,d)$ transformation in string theory or supergravity is T-duality. For a manifold with n isometries, the T-duality group is given by $O(n,n)$ and it can be embedded trivially in the $O(d,d)$ group acting here. In particular, the T-duality group element corresponding to Buscher rules is given by

$$O_T = \left(\begin{array}{cc|cc} 0_n & & \mathbb{I}_n & \\ & \mathbb{I}_{d-n} & & 0_{d-n} \\ \hline \mathbb{I}_n & & 0_n & \\ & 0_{d-n} & & \mathbb{I}_{d-n} \end{array} \right) . \tag{2.3.16}$$

Even if $O(d,d)$ preserves the metric η, the structure group of E is only a subgroup $G_{\text{geom}} \subset O(d,d)$. It is given by the patching conditions (2.3.6), so it is the semi-direct product $G_{\text{geom}} = GL(d) \ltimes G_B$, generated by the $GL(d)$ transformations and the B-transforms. The embedding of $G_{\text{geom}} \subset O(d,d)$ is fixed by the projection $\pi : E \to TM$. It is the subgroup which leaves the image of the related embedding $T^*M \to E$ invariant.

GCG, via the generalized tangent bundle E, gives a geometric picture of the whole NSNS sector of supergravity (we will discuss the dilaton further on). In addition, it incorporates naturally the T-duality action in a covariant way. These are arguments in favour of the use of this formalism in supergravity.

2.3.2 Generalized metric and vielbein

In addition to η, one can introduce the generalized metric on E which combines the metric g on M and the B-field into a single object

$$\mathcal{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} . \tag{2.3.17}$$

One way to justify this definition is to introduce a split of the bundle E into two orthogonal d-dimensional sub-bundles $E = C_+ \oplus C_-$ such that the metric η decomposes into a positive-definite metric on C_+ and a negative-definite metric on C_-. The two sub-bundles are defined as

$$C_\pm = \{V \in TM \oplus T^*M \quad : \quad V_\pm = v + (B \pm g)v\} , \tag{2.3.18}$$

and have a natural interpretation in string theory compactified in a six-dimensional manifold as the right and left mover sectors. Then the generalized metric is defined by

$$\mathcal{H} = \eta|_{C_+} - \eta|_{C_-} . \tag{2.3.19}$$

The gluing conditions on the double overlaps for the metric and B-field are

$$g_{(\alpha)} = g_{(\beta)}, \qquad B_{(\alpha)} = B_{(\beta)} - \mathrm{d}\Lambda_{(\alpha\beta)} . \tag{2.3.20}$$

A length (square) element would be given by $\mathrm{d}X^T\mathcal{H}\mathrm{d}X$, where

$$\mathrm{d}X^M = \begin{pmatrix} \mathrm{d}y^m \\ \partial_m \end{pmatrix} . \tag{2.3.21}$$

The generalized metric $\mathcal{H}$ transforms under $O(d,d)$ as

$$\mathcal{H} \mapsto \mathcal{H}' = O^T\mathcal{H}O . \tag{2.3.22}$$

Actually expression (2.3.17) is well known from the study of T-duality, where it parametrizes the moduli of d-dimensional toroidal compactifications, and indeed its transformation under $O(d,d)$ is the same as in standard T-duality $O(n,n)$.

We can also introduce generalized vielbein $\mathcal{E}$. They parametrize the coset $O(d,d)/O(d) \times O(d)$, where the local $O(d) \times O(d)$ transformations play the same role as the local Lorentz symmetry for ordinary vielbein. There are many different conventions one could use to define the generalized vielbein, which are connected by local frame transformations. Here we define them as

$$\mathcal{E} = \begin{pmatrix} e & 0 \\ -e^{-T}B & e^{-T} \end{pmatrix} , \tag{2.3.23}$$

where e are the ordinary vielbein, $e^T\ e = g$, and the $\mathcal{E}$ are such that

$$\eta = \mathcal{E}^T \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \mathcal{E}, \qquad \mathcal{H} = \mathcal{E}^T \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \mathcal{E} . \tag{2.3.24}$$

In terms of indices, it is clear that the generalized vielbein are written as $\mathcal{E}^A{}_M$, where A are generalized frame indices, while M are the space E indices.

The generalized vielbein parametrize the coset $O(d,d)/O(d)\times O(d)$. From the previous expressions, it is clear that the $O(d,d)$ transformation O acts on the right, while the $O(d) \times O(d)$ acts on the left:

$$\mathcal{E} \mapsto \mathcal{E}' = K\mathcal{E}O \ , \ O \in O(d,d) \ , \ K \in O(d) \times O(d) . \tag{2.3.25}$$

Note that the choice of generalized vielbein (2.3.23) is invariant under the $G_{\rm geom}$ subgroup of $O(d,d)$ transformations. Furthermore, note that the B-field enters in $\mathcal{E}$ exactly as an ordinary connection one-form would enter, in the off-diagonal part, and transforms under $G_{\rm geom}$ as a connection would, with a gauge transformation.

One can a priori choose a different set of vielbein for the left and right mover sectors, or equivalently for $C_\pm$

$$g = e_\pm^T e_\pm \quad \text{or} \quad g_{mn} = e^a_{\pm m} e^b_{\pm n}\delta_{ab} \ , \tag{2.3.26}$$
$$g^{-1} = \hat{e}_\pm \hat{e}_\pm^T \quad \text{or} \quad g^{mn} = \hat{e}^m_{\pm a}\hat{e}^n_{\pm b}\delta^{ab} \ , \tag{2.3.27}$$

and $e_\pm \hat{e}_\pm = \hat{e}_\pm e_\pm = \mathbb{I}$. Each of the two sets is acted upon by one of the local $O(d)$ groups. The expression for the generalized vielbein then becomes

$$\mathcal{E} = \frac{1}{2}\begin{pmatrix} (e_+ + e_-) + (\hat{e}_+^T - \hat{e}_-^T)B & (\hat{e}_+^T - \hat{e}_-^T) \\ (e_+ - e_-) - (\hat{e}_+^T + \hat{e}_-^T)B & (\hat{e}_+^T + \hat{e}_-^T) \end{pmatrix} \ . \tag{2.3.28}$$

Since the supergravity spinors transform under one or the other of the $O(d)$ groups, it is natural to use the local frame transformations to set $e_+ = e_-$ so that the same spin-connections appear, for instance, in the derivatives of the two gravitini. Explicitly, the $O(d)\times O(d)$ action has the form

$$\mathcal{E} \mapsto K\mathcal{E} \ , \qquad K = \frac{1}{2}\begin{pmatrix} O_+ + O_- & O_+ - O_- \\ O_+ - O_- & O_+ + O_- \end{pmatrix} \ , \tag{2.3.29}$$

where $O_\pm$ are the $O(d)$ transformation acting on the vielbein $e_\pm$. With this choice the generalized vielbein reduce to those in (2.3.23). Notice that locally it is always possible to put the generalized vielbein in standard lower triangular form (2.3.23). A different issue is whether this can be done globally. In [27], such a distinction has been used to propose a criterion for non-geometry [27]. Suppose that for a given background the generalized vielbein $\mathcal{E}$ has the generic structure (2.3.28) (T-duality for instance gives vielbein of this form). Then if the transformation (2.3.29) bringing a generic $\mathcal{E}$ into the lower triangular form is not globally defined the corresponding background is non-geometric. See appendix C.2 for an illustration.

2.3.3 Spinors in GCG

One can define spinors on E. Given the metric η, the Clifford algebra on E is Cliff(d,d)

$$\{\Gamma^m, \Gamma^n\} = \{\Gamma_m, \Gamma_n\} = 0 \ , \quad \{\Gamma^m, \Gamma_n\} = \delta^m_n \ , \tag{2.3.30}$$

with $m,\ n = 1\ldots d$. The *Spin*(d,d) spinors are Majorana–Weyl. The positive and negative chirality spin bundles, $S^\pm(E)$, are isomorphic to even and odd forms on E. It is easy to see that, locally, the Clifford action of $V = (v,\xi) \in E$ on the spinors can indeed be realized as an action on forms

$$V\cdot\Psi = (v^m\Gamma_m + \xi_m\Gamma^m)\Psi = \iota_v\Psi + \xi\wedge\Psi \ , \tag{2.3.31}$$

with $\Gamma^n = {\rm d}x^n\wedge$ and $\Gamma_m = \iota_{\partial_m}$. One gets

$$(V_1V_2 + V_2V_1)\cdot\Psi = 2\eta(V_1,V_2)\Psi \ , \tag{2.3.32}$$

as required. Therefore, these $O(d,d)$ spinors can be understood as polyforms, i.e. sums of forms of different degrees

$$\Psi_\pm \in L\otimes \Lambda^{\rm even/odd}T^*M\Big|_{U_\alpha} \ . \tag{2.3.33}$$

The isomorphism is determined by the trivial line bundle L, whose sections are given in terms of the 10-dimensional dilaton, $e^{-\phi} \in L$. L is needed in order for the spinors to transform correctly[6] under $GL(d)$.

The spinors are not globally defined on E. At the overlap of two patches they transform as

$$\Psi^{\pm}_{(\alpha)} = e^{-\mathrm{d}\Lambda_{(\alpha\beta)}} \Psi^{\pm}_{(\beta)} \,. \tag{2.3.35}$$

We define the $O(d,d)$ generators in the spinorial representation as

$$\sigma^{MN} = [\Gamma^M, \Gamma^N] \,, \tag{2.3.36}$$

with M, N $d+d$ indices. Then the group element in the spinorial representation is

$$O = e^{-\frac{1}{4}\Theta_{MN}\sigma^{MN}} \,. \tag{2.3.37}$$

The matrix Θ_{MN} is antisymmetric and reads

$$\Theta_{MN} = \begin{pmatrix} a^m{}_n & \beta^{mn} \\ B_{mn} & -a_m{}^n \end{pmatrix} \,, \tag{2.3.38}$$

where $a^m{}_n$, B_{mn} and β^{mn} parametrize the generators of the $GL(d)$ transformations, B-transform and β-transform, respectively. In particular, the $GL(d)$ action is given by [7]

$$\begin{aligned} O_a &= e^{-\frac{1}{4}(a^m{}_n[\Gamma_m,\Gamma^n] - a_m{}^n[\Gamma^m,\Gamma_n])} \\ &= e^{-\frac{1}{2}\mathrm{Tr}(a) + a^m{}_n \mathrm{d}x^n \wedge\, \iota_{\partial_m}} \\ &= \frac{1}{\sqrt{\det A}} \; e^{a^m{}_n \mathrm{d}x^n \wedge\, \iota_{\partial_m}} \,. \end{aligned} \tag{2.3.39}$$

Similarly, for B and β-transforms, we obtain

$$O_B = e^{-\frac{1}{2}B_{mn}\Gamma^{mn}} = e^{-\frac{1}{2}B_{mn}\mathrm{d}x^m\wedge \mathrm{d}x^n\wedge} \,, \tag{2.3.40}$$

$$O_\beta = e^{-\frac{1}{2}\beta^{mn}\Gamma_{mn}} = e^{-\frac{1}{2}\beta^{mn}\iota_m\llcorner\, \iota_n\llcorner} , \tag{2.3.41}$$

that we write loosely as

$$O_B = e^{-B} \,,\; O_\beta = e^{-\beta\llcorner} \,. \tag{2.3.42}$$

The last three transformations are actually connected to the identity. Therefore, working out their spinorial representation is easy. Indeed, in principle, they can be written as an exponential, so one can read the parameters of the transformation Θ_{MN} out of it. It is not the case of the T-duality, as one can see from (2.3.16). Nevertheless, a spinorial operator can still be worked out, and according to [28, 29, 27], it is given by

$$T = \mathrm{d}t \wedge + \iota_t\llcorner \,, \tag{2.3.43}$$

for a T-duality in the t direction.

Elements connected to the identity are $Spin(d,d)$ elements. As we act on complex spinors, if one wants to perform a general $Pin(d,d)$ transformation, one could also allow for a phase transformation

$$O^{\pm}_c = e^{i\theta^{\pm}_c} O \,, \tag{2.3.44}$$

where the indices $\pm$ will used further when acting on $\Psi_\pm$. This phase can be can understood equivalently the following way. The isomorphism (2.3.33) is defined up to a multiplication by a complex

[6] The dilaton transforms under $O(d,d)$ as

$$e^{\phi'} = e^{\phi} \left[\frac{\det(g')}{\det(g)}\right]^{\frac{1}{4}} \,. \tag{2.3.34}$$

number and, in general, defines line bundles of spinors (the line L can be understood as complex, i.e. does not contain only the dilaton). On symplectic or complex manifolds, this line bundle can be trivialized, so we can fix this phase and have global spinors (these are manifolds with vanishing first Chern-class). In general this condition is not satisfied. When transforming the spinors, one could a priori change the section of this complex line bundle L, and so one can allow for a phase as in (2.3.44). By a slight abuse of language, we will refer to the lines of spinors as simply spinors.

2.4 Pure spinors of GCG and SUGRA vacua

An important observation that allows to connect GCG with supersymmetry compactifications in string theory is that Cliff(6,6) pure spinors can be written as tensor product of Cliff(6) spinors. In this section we will see that it is possible to translate the topological requirement of having globally defined spinors on M to the requirement that the structure group on E must be $SU(3) \times SU(3)$. Similarly the closure of the SUSY spinors is equivalent to a set of differential equations on the pure spinors.

2.4.1 Internal spinors and GCG pure spinors

Let us first consider spinors on $TM \oplus T^*M$. As discussed, these are Majorana-Weyl Cliff(d,d) spinors, and they can be seen as polyforms: sums of even/odd differential forms, which correspond to positive/negative chirality spinors. We will be interested in pure spinors. These are vacua of the Cliff(d,d): a Cliff(d,d) spinor is pure if it is annihilated by half of the Cliff(d,d) gamma matrices.

Cliff(6,6) pure spinors on $TM \oplus T^*M$ can be obtained as tensor products of Cliff(6) spinors, since bispinors are isomorphic to forms via the Clifford map

$$C = \sum_k \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \quad \leftrightarrow \quad C = \sum_k \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} \gamma^{i_1 \ldots i_k} \ , \tag{2.4.1}$$

and in six dimensions, any Cliff(6) spinor is pure.

In the supergravity context, it is therefore natural to define the Cliff(6,6) pure spinors on $TM \oplus T^*M$ as a bi-product of the internal supersymmetry parameters

$$\begin{aligned} \Phi_+ &= \eta^1_+ \otimes \eta^{2\dagger}_+ \ , \\ \Phi_- &= \eta^1_+ \otimes \eta^{2\dagger}_- \ . \end{aligned} \tag{2.4.2}$$

They can be seen as polyforms via the Fierz identity

$$\eta^1_+ \otimes \eta^{2\dagger}_\pm = \frac{1}{8} \sum_{k=0}^{6} \frac{1}{k!} \left(\eta^{2\dagger}_\pm \gamma_{\mu_k \ldots \mu_1} \eta^1_+ \right) \gamma^{\mu_1 \ldots \mu_k} \ . \tag{2.4.3}$$

The explicit expressions of the two pure spinors depend on the form of the spinors η^1 and η^2. We choose to parametrize these spinors as

$$\begin{aligned} \eta^1_+ &= a\eta_+ \ , \\ \eta^2_+ &= b(k_{||}\eta_+ + k_\perp \frac{z\eta_-}{2}) \ . \end{aligned} \tag{2.4.4}$$

η_+ and $\chi_+ = \frac{1}{2} z\eta_-$ in (2.4.4) define an $SU(2)$ structure (see section 2.2.2). $k_{||}$ is real and $0 \leq k_{||} \leq 1$, $k_\perp = \sqrt{1 - k_{||}^2}$. a and b are complex numbers related to the norms of the spinors η^i_+

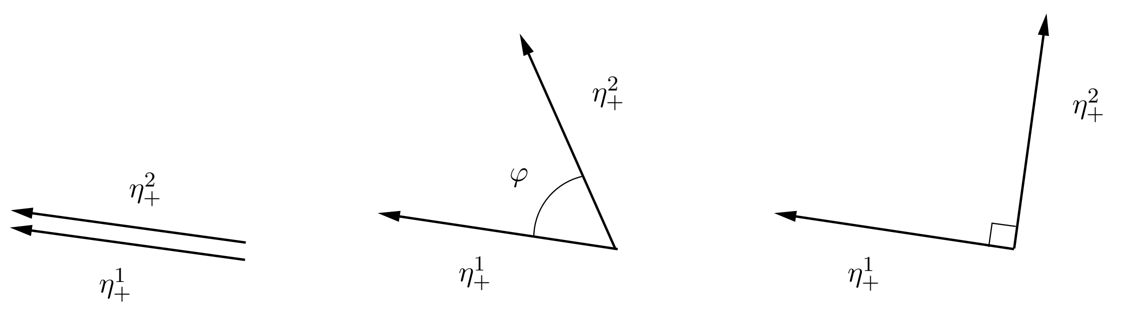
$$a = ||\eta^1_+||\ e^{i\alpha} \ , \qquad b = ||\eta^2_+||\ e^{i\beta} \ . \tag{2.4.5}$$

In the following, we will always take $|a| = |b|$, so that $||\eta_+^1|| = ||\eta_+^2||$. This condition is implied by the presence of supersymmetric sources, or equivalently by the orientifold projection [30, 29].

Depending on the values of the parameters $k_{||}$ and $k_\perp$, one can define different G-structures on the internal manifold. $k_{||}$ and $k_\perp$ can be related to the "angle" between the spinors. Let us introduce the angle φ

$$k_{||} = \cos(\varphi),\ \ k_\perp = \sin(\varphi),\ \ 0 \leq \varphi \leq \frac{\pi}{2}\ . \tag{2.4.6}$$

For $k_\perp = 0$, the spinors become parallel, so there is only one globally defined spinor, and this corresponds to an $SU(3)$ structure. When $k_\perp \neq 0$, the two spinors are genuinely independent, so we get an $SU(2)$ structure [31]. In that case, we will need to further distinguish the two cases $k_{||} = 0$ and $k_{||} \neq 0$. We call them orthogonal $SU(2)$ structure and intermediate $SU(2)$ structure respectively, in reference to the angle between the spinors. We get the following pictures[7]:



$SU(3)$ structure: $k_{||} = 1,\ k_\perp = 0$ | Intermediate $SU(2)$ structure: $k_{||} \neq 0,\ k_\perp \neq 0$ | Orthogonal $SU(2)$ structure: $k_{||} = 0,\ k_\perp = 1$

Figure 2.1: The different structures

Given the parametrization (2.4.4) of the internal spinors, we can obtain explicit expressions for the pure spinors as polyforms [32]

$$\begin{aligned}\Phi_+ &= \frac{|a|^2}{8} e^{i\theta_+} e^{\frac{1}{2} z\wedge\overline{z}} (k_{||} e^{-ij} - i k_\perp \omega)\ ,\\ \Phi_- &= -\frac{|a|^2}{8} e^{i\theta_-} z \wedge (k_\perp e^{-ij} + i k_{||}\omega)\ ,\end{aligned} \tag{2.4.9}$$

where the forms are defined in section 2.2.2 and the phases $\theta_\pm$ are related to the phases of the spinors η^i: $\theta_+ = \alpha - \beta$, $\theta_- = \alpha + \beta$.

A pure spinor can always be written as [7] $\omega_k \wedge e^{\tilde{b} + i\tilde{\omega}}$ where ω_k is a holomorphic k-form, and $\tilde{b}$ and $\tilde{\omega}$ are real two-forms. The rank k of ω_k is called the type of the spinor. For the intermediate $SU(2)$ structure where both $k_{||}$ and $k_\perp$ are non-zero, it is possible to "exponentiate" ω and get from (2.4.9)

$$\begin{aligned}\Phi_+ &= \frac{|a|^2}{8} e^{i\theta_+} k_{||}\ e^{\frac{1}{2} z\wedge\overline{z} - ij - i\frac{k_\perp}{k_{||}}\omega}\ ,\\ \Phi_- &= -\frac{|a|^2}{8} e^{i\theta_-} k_\perp\ z \wedge e^{-ij + i\frac{k_{||}}{k_\perp}\omega}\ ,\end{aligned} \tag{2.4.10}$$

[7] As a comparison to (2.2.19), one can work out the embedding of the defined $SU(2)$ structure in the $SU(3)$ structure defined by $\frac{\eta_+^2}{||\eta_+^2||}$ ($\tilde{J}$ and $\tilde{\Omega}$). It is given by the previous $U(1)$ parameter φ [31]:

$$\tilde{J} = \cos(2\varphi) j + \frac{i}{2} z \wedge \overline{z} + \sin(2\varphi)\,\mathrm{Re}(\omega)\ , \tag{2.4.7}$$

$$\tilde{\Omega} = -\sin(2\varphi) z \wedge j + z \wedge (\cos(2\varphi)\,\mathrm{Re}(\omega) + i\,\mathrm{Im}(\omega))\ . \tag{2.4.8}$$

so that the spinors have definite types: 0 and 1. In the case of the $SU(3)$ structure ($k_\perp = 0$), we get that pure spinors are of type 0 and 3

$$\Phi_+ = \frac{|a|^2}{8} e^{i\theta_+} e^{-iJ} , \qquad \Phi_- = -ie^{i\theta_-} \frac{|a|^2}{8} \Omega , \tag{2.4.11}$$

while in the case of the orthogonal $SU(2)$ structure ($k_{||} = 0$), the types are 1 and 2:

$$\Phi_+ = -i\frac{|a|^2}{8} e^{i\theta_+} \omega \wedge e^{\frac{1}{2} z\wedge\bar{z}} , \qquad \Phi_- = -\frac{|a|^2}{8} e^{i\theta_-} z \wedge e^{-ij} . \tag{2.4.12}$$

If a pure spinor is closed, its type k serves as a convenient way of characterising the geometry. The manifold then admits a complex structure along $2k$ real directions, and a symplectic structure along $d - 2k$ directions.

So far, we discussed $\Phi_\pm$ defined in (2.4.2): these are pure spinors on $TM \oplus T^*M$ [7]. Out of these, one can get pure spinors on E. The latter are the proper ones to act on with an $O(d, d)$ transformation, as discussed in the previous section. To get them, one should insert the proper dependence in B-field (brings to E) and dilaton (gives the line bundle L). So we define the "normalized" pure spinors[8] $\Psi_\pm$ on E as

$$\Psi_\pm = 8\ e^{-\phi} e^{-B} \frac{\Phi_\pm}{||\Phi_\pm||} , \tag{2.4.13}$$

where we define the norm of a polyform via the Mukai pairing[9], as $8\langle\Phi_\pm, \overline{\Phi}_\pm\rangle = -i||\Phi_\pm||^2 V$, and V is the volume form. Out of (2.4.2), we get $\left\langle \Phi_\pm, \overline{\Phi}_\pm \right\rangle = -\frac{i}{8}||\eta^1_+||^2||\eta^2_\pm||^2 V$, i.e. $||\Phi_\pm|| = |ab|$. So we get for instance for an $SU(3)$ structure

$$\Psi_+ = e^{i\theta_+} e^{-\phi} e^{-B}\, e^{-iJ} , \tag{2.4.15}$$
$$\Psi_- = -ie^{i\theta_-} e^{-\phi} e^{-B} \Omega . \tag{2.4.16}$$

The existence of a pure spinor reduces the structure group of E from $O(d, d)$ to $U(d/2, d/2)$. If the associated line bundle of complex differential forms can be trivialized, the structure group is reduced to $SU(d/2, d/2)$. It can be further reduced in presence of a second compatible pure spinor. Two pure spinors are said to be compatible when they have $d/2$ common annihilators. This can be rephrased in a set of compatibility conditions the spinors must satisfy. Given the action of a generalized vector on a polyform (2.3.31), the compatibility conditions of two pure spinors Φ_1 and Φ_2 read

$$\begin{aligned} &\left\langle \Phi_1, \overline{\Phi}_1 \right\rangle = \left\langle \Phi_2, \overline{\Phi}_2 \right\rangle \neq 0 , \\ &\langle \Phi_1, V \cdot \Phi_2 \rangle = \left\langle \overline{\Phi}_1, V \cdot \Phi_2 \right\rangle = 0,\ \forall\ V\ \in\ TM \oplus T^*M . \end{aligned} \tag{2.4.17}$$

The existence of a second compatible pure spinor reduces the structure group from $U(d/2, d/2)$ to $U(d/2) \times U(d/2)$ (or equivalently from $SU(d/2, d/2)$ to $SU(d/2) \times SU(d/2)$ when the line bundles of the complex differential forms can be trivialized).

In our case, the pure spinors $\Phi_\pm$ (2.4.2) are compatible, so such a pair defines an $SU(3) \times SU(3)$ structure on $TM \oplus T^*M$. Depending on the relation between the spinors $\eta^{1,2}_+$, this translates on TM into the $SU(3)$, orthogonal $SU(2)$ or intermediate $SU(2)$ structures discussed above. So the formalism of GCG provides a unified topological requirement for the manifold M: for a $\mathcal{N} = 1$

[8] They are also called twisted pure spinors, because of the B-field.

[9] The Mukai pairing for two polyforms Z_i is defined as

$$\langle Z_1, Z_2\rangle = (Z_1 \wedge \lambda(Z_2))_{\text{top}} , \tag{2.4.14}$$

where top selects the top-form, and λ has been defined in (2.2.3).

vacuum, one should find on $TM \oplus T^*M$ an $SU(3) \times SU(3)$ structure (equivalent to having a pair of internal spinors).

In practice, we will verify that our vacua admit a pair of compatible pure spinors. One can actually show (see appendix A.2.2) that the "wedge" structure conditions (2.2.14), or (2.2.16) and (2.2.17), imply the compatibility conditions in any of the three cases, so one can verify these conditions instead.

2.4.2 SUSY conditions

The SUSY conditions are given by the annihilation of the fermionic variations (2.2.8) and (2.2.9) of type II SUGRA. According to the decomposition of the ten-dimensional supersymmetry parameters (2.2.11) and (2.2.12), into a four and six dimensional factors, we can split the SUSY variations into external ($4d$) and internal ($6d$) components. In [5] it was shown that such a system of equations can be rewritten as a set of differential conditions on the pair (2.4.2) of compatible pure spinors

$$(\mathrm{d} - H\wedge)(e^{2A-\phi}\Phi_1) = 0 \ , \tag{2.4.18}$$

$$(\mathrm{d} - H\wedge)(e^{A-\phi}\,\mathrm{Re}(\Phi_2)) = 0 \ , \tag{2.4.19}$$

$$(\mathrm{d} - H\wedge)(e^{3A-\phi}\,\mathrm{Im}(\Phi_2)) = \frac{|a|^2}{8} e^{3A} * \lambda(F) \ , \tag{2.4.20}$$

with λ defined in (2.2.3), and with

$$\Phi_1 = \Phi_\pm \ , \qquad \Phi_2 = \Phi_\mp \ , \tag{2.4.21}$$

for IIA/IIB (upper/lower). Later on, we will take $|a|^2 = e^A$. These SUSY conditions generalize the Calabi-Yau condition for fluxless compactifications. Indeed, the first of these equations implies that one of the two pure spinors (the one with the same parity as the RR fields) must be twisted (because of the $-H\wedge$) conformally closed. A manifold admitting a twisted closed pure spinor is a twisted Generalized Calabi-Yau (GCY, see the precise definition in [6, 7] or [29]). So we will look for vacua on such manifolds.

We recall from section 2.2.1 that the SUSY conditions and the BI imply together the e.o.m. (for Minkowski). For intermediate $SU(2)$ structures (for which $\frac{k_\perp}{k_{||}}$ is constant) in the large volume limit, we will get from our SUSY conditions that the H BI is automatically satisfied. So only the RR BI will have to be checked.

2.5 Outline of the thesis

In chapter 1 and in this chapter, we motivated our search of supersymmetric vacua of ten-dimensional type II SUGRA and we showed that Generalized Complex Geometry provides a natural formalism to use to study $\mathcal{N} = 1$ flux vacua. Here is now the outline of the rest of the thesis.

Chapter 3 discusses Minkowski supersymmetric ten-dimensional solutions on solvmanifolds (twisted tori). These manifolds are interesting candidates for such solutions (some subclasses of them are proved to be GCY), and also provide an easy set-up to explicitly solve the SUSY equations for the pure spinors and Bianchi identities for the fluxes. After a brief review of their geometric properties (a more detailed account is provided in appendix B.1), we present the resolution method to look for solutions, and give a list of known solutions on these manifolds. Then we focus on a particular type of solutions: those admitting an intermediate $SU(2)$ structure. In order to find them, we have to adapt slightly the method described, and introduce a particular basis of forms, which simplifies the orientifold projection conditions, and the SUSY conditions. Appendix B.2 contains details on these points. Then, we present three solutions found. By taking a limit on these solutions, we are able to recover known solutions with either $SU(3)$ or orthogonal $SU(2)$ structure, and also find a new one. Finally, we derive conditions for intermediate $SU(2)$ structure solutions to be β-transforms of an $SU(3)$ solution, and show it is the case for one of the solutions found. This chapter is based on the paper [1] and further unpublished work.

Chapter 4 deals with a particular type of $O(d,d)$ transformation, named the twist, that can be used to generate new supersymmetric solutions on Minkowski. The idea is first to propose a transformation which constructs the one-forms of a solvmanifold out of those of a torus. This transformation is then embedded and extended in GCG as a local $O(d,d)$ transformation, to relate solutions on torus to solutions on nil- and solvmanifolds. The conditions to generate new supersymmetric solutions are discussed and used to recover known solutions on nilmanifolds and find a new one on a solvmanifold. We also present a new, fully localized, solution on a solvmanifold, and discuss the possibility of obtaining non-geometric backgrounds out of the twist. Finally, we discuss how to write the SUSY conditions for the heterotic string in terms of pure spinors, and then use the twist to relate some torsional solutions found in the literature. In the associated appendix C, we come back to the definition of the twist as a way to construct one-forms, give a list of solvmanifolds, discuss possible non-geometric T-duals of solvmanifold solutions, and explain a possible extension of the local $O(d,d)$ transformation to the gauge bundle of heterotic string. This chapter is based on the two papers [2] and [3].

Chapter 5 discusses the possibility to obtain non-SUSY solutions on a four-dimensional de Sitter space. The main motivation for such solutions is cosmological, as discussed in chapter 1. We first explain what are the major difficulties in obtaining supergravity solutions with a positive cosmological constant. Then we propose an ansatz for SUSY breaking sources, which can help to lift the value of Λ. This ansatz is based on the idea of preserving some sort of first order equations based on the $SU(3)$ structure, despite the breaking of SUSY conditions. This idea has already been used to find non-SUSY solutions. We then provide an example of a de Sitter solution where the internal manifold is a solvmanifold. This solution can be understood as a deviation from the new SUSY solution found on a solvmanifold in chapter 4. We also discuss the possibility of generalize the ansatz to find a first order formalism which would imply, together with BI, the e.o.m., as the SUSY conditions do in the SUSY case. We end the chapter by a partial four-dimensional analysis of the stability of the solution found. This chapter is based on the paper [3].

In chapter 6, we summarize the results of this thesis, and present further ideas to be studied.

Chapter 3

Solutions on solvmanifolds

3.1 Introduction

In the previous chapter, we showed how Generalized Complex Geometry provides an interesting set-up where to study supersymmetric flux vacua of type II SUGRA, where the ten-dimensional space-time is split in four-dimensional Minkowski and a six-dimensional internal manifold M. In particular, in order to have $\mathcal{N} = 1$ Minkowski flux vacua, the internal space must be a Generalized Calabi-Yau (GCY) manifold.

In this chapter, we discuss whether one can provide explicit examples of GCY flux backgrounds. The simplest examples of flux vacua consist of a warped CY in type IIB (in the simplest case a warped T^6) with $O3$-plane and imaginary self-dual three-form flux. Then a successful approach [33] to produce new flux vacua is to T-dualise the warped CY solutions. The resulting manifolds are twisted tori, i.e. fibrations of circles over torus base. Mathematically, these are solvmanifolds: manifolds constructed out of particular Lie groups called solvable groups. A subset is given by nilpotent groups, out of which one gets nilmanifolds. The latter have been proved to be all GCY [23], and indeed, some SUSY vacua on them have been found via T-duality.

Using Generalized Complex Geometry, instead of using dualities, one can actually try to find vacua on a GCY (like one of the nilmanifolds) by directly solving the SUSY constraints and Bianchi identities for the fluxes. This is the strategy used in [29] to determine flux vacua on nil- and (some) solvmanifolds. The authors recovered some known solutions, that had been already obtained by T-duality from a conformal CY, but also found new, non T-dual, flux vacua. These solutions were obtained after performing a scan on all six-dimensional nilmanifolds, and among some solvmanifolds.

A scan was possible because the resolution method on these manifolds is rather algorithmic. In particular, the manifolds considered are parallelizable so they provide through their Maurer-Cartan forms a six-dimensional basis of globally defined one-forms. Out of these one-forms, one can construct pure spinors and try to tune their free parameters to get a solution.

Before giving more details on the resolution method and the solutions found, let us first review briefly the geometric properties of nil- and solvmanifolds. We will be interested in these manifolds in the rest of this thesis. The rest of the chapter will be dedicated to a particular type of solutions, those admitting intermediate $SU(2)$ structure. To find such solutions, one has to adapt slightly the resolution method. We will then give three solutions found, and discuss some relations between them and solutions with $SU(3)$ or orthogonal $SU(2)$ structure.

3.2 Nil- and solvmanifolds

In this thesis we are interested in string backgrounds where the internal manifold is a nil- or a solv-manifold. We give in this section an account on their geometric properties.

Nil- and solvmanifolds are homogeneous spaces constructed from nilpotent or solvable groups G, nilpotent being actually a particular case of solvable. When the group G is not compact, the manifold can be made compact by quotienting G by a lattice Γ, i.e. a discrete co-compact subgroup of G. The dimension of the resulting manifold is the same as that of the group[1] G. Here we will focus on manifolds of dimension six. It can be proven [34] that a lattice Γ can always be found for nilpotent groups, while for generic solvable ones its existence is harder to establish. We refer to appendix B.1 for a detailed discussion of the algebraic aspects and the compactness properties of nil- and solvmanifolds. Here, we focus on their geometry.

Given a d-dimensional Lie algebra $\mathfrak{g}$ expressed in some vector basis $\{E_1, \ldots, E_d\}$ as

$$[E_b, E_c] = f^a{}_{bc} E_a \;, \tag{3.2.1}$$

where $f^a{}_{bc}$ are the structure constants, we can define the dual space of one-forms $\mathfrak{g}^*$ with basis $\{e^1, \ldots, e^d\}$. They satisfy the Maurer-Cartan equation

$$\mathrm{d}e^a = -\frac{1}{2} f^a{}_{bc} e^b \wedge e^c = -\sum_{b<c} f^a{}_{bc}\, e^b \wedge e^c \;, \tag{3.2.2}$$

with the exterior derivative d. Since $\mathfrak{g}^* \approx T_e G^*$, $\{e^1, \ldots, e^d\}$ provide, by left invariance, a basis for the cotangent space $T_x G^*$ at every point $x \in G$ and, thus, are globally defined one-forms on the manifold. When the manifold is obtained as a quotient with a lattice Γ, the one-forms will have non-trivial identification through the lattice action[2]. Nil- and solvmanifolds, as we define them here, are always parallelizable [36], even if they are not necessarily Lie groups.

The Maurer-Cartan equations reflect the topological structure of the corresponding manifolds. For example, nilmanifolds all consist of iterated fibrations of circles over tori, where the iterated structure is related to the descending or ascending series of the algebra (see [23, 37, 38]). This can be easily seen on a very simple example, the three-dimensional nilmanifold obtained from the three-dimensional Heisenberg algebra

$$[E_2, E_3] = E_1 \qquad \Leftrightarrow \qquad \mathrm{d}e^1 = -e^2 \wedge e^3 \,. \tag{3.2.3}$$

The Maurer-Cartan equation is solved by the one-forms

$$e^1 = \mathrm{d}x^1 - x^2 \mathrm{d}x^3 \;,\; e^2 = \mathrm{d}x^2,\; e^3 = \mathrm{d}x^3 \;. \tag{3.2.4}$$

From the connection form, $-x^2\mathrm{d}x^3$, one can read the topology of the corresponding nilmanifold $\mathcal{M}$, which is a non-trivial fibration of the circle in direction 1 on the two-torus in directions $2, 3$:

$$\begin{array}{ccc} S^1_{\{1\}} & \hookrightarrow & \mathcal{M} \\ & & \downarrow \\ & & T^2_{\{23\}} \end{array} \tag{3.2.5}$$

A nilpotent group is particular case of a solvable group for which the question of compactness (existence of a lattice) and then the topology of the corresponding nilmanifold (iterated fibrations of circles) are simpler than for generic solvable groups. Let us now focus on non-nilpotent solvable

[1] This definition of solvmanifold it is not the most general: one could consider cases where the d-dimensional solvmanifold is the quotient of a higher dimensional group with a continuous subgroup Γ. This is the case for the Klein bottle, for instance.

[2] In general there is a natural inclusion $(\Lambda\mathfrak{g}^*, \delta) \to (\Lambda(G/\Gamma), d)$ between the Chevalley-Elienberg complex on G and the de Rham complex of differential forms on G/Γ. This inclusion induces an injection map between cohomology groups $H^*(\mathfrak{g}) \to H^*_{dR}(G/\Gamma)$ which turns out to be an isomorphism for completely solvable groups. We recall that a Lie group G with Lie algebra $\mathfrak{g}$ is said to be completely solvable if the linear map $ad_X : \mathfrak{g} \to \mathfrak{g}$ only has real roots $\forall X \in \mathfrak{g}$. Note that all nilmanifolds are completely solvable and thus the injection is an isomorphism (Nomizu's theorem [35]), the extension to non-nilpotent completely solvable groups being the so-called Hattori theorem [36]. For more details and for a list of Betti numbers of solvmanifolds up to dimension six see [37].

groups.

Solvable groups are classified according to the dimension of the nilradical $\mathfrak{n}$ (the largest nilpotent ideal) of the corresponding algebra. In six dimensions, $\mathfrak{n}$ can have dimension from 3 to 6. If dim $\mathfrak{n} = 6$, then $\mathfrak{n} = \mathfrak{g}$ and the algebra is nilpotent. At the level of the group[3] we have that, if dim$N < 6$, then G contains an abelian subgroup of dimension k [39, 40]. This means we have $G/N = \mathbb{R}^k$. When the group admits a lattice Γ, one can show that $\Gamma_N = \Gamma \cap N$ is a lattice in N, $\Gamma N = N\Gamma$ is a closed subgroup of G, and so $G/(N\Gamma) = T^k$ is a torus. The solvmanifold is a non-trivial fibration of a nilmanifold over the torus T^k

$$\begin{array}{ccc} N/\Gamma_N = (N\Gamma)/\Gamma & \hookrightarrow & G/\Gamma \\ & & \downarrow \\ & & T^k = G/(N\Gamma) \end{array} \tag{3.2.6}$$

This bundle is called the Mostow bundle [41]. As we shall see, the corresponding fibration can be more complicated than in the nilmanifold case. In general, Mostow bundles are not principal.

Let us focus on a particular case of non-nilpotent solvable groups, called almost abelian solvable groups, for which the construction of the Mostow bundle is particularly simple. Consider first almost nilpotent solvable groups. These are solvable groups that have nilradical of dimension dim $N =$ dim $G - 1$. As discussed in appendix B.1.1, the group is then given by the semi-direct product

$$G = \mathbb{R} \ltimes_{\mu} N \tag{3.2.7}$$

of its nilradical with $\mathbb{R}$, where μ is some action on N depending on the direction $\mathbb{R}$

$$(t_1, n_1) \cdot (t_2, n_2) = (t_1 \cdot t_2, n_1 \cdot \mu_{t_1}(n_2)) \qquad \forall (t, n) \in \mathbb{R} \times N\ . \tag{3.2.8}$$

In general, we label by t the coordinate on $\mathbb{R}$ and by ∂_t the corresponding vector of the algebra. From a geometrical point of view, $\mu(t)$ encodes the fibration of the Mostow bundle.

An almost abelian solvable group is an almost nilpotent group whose nilradical is abelian

$$N = \mathbb{R}^{\dim G - 1}\,. \tag{3.2.9}$$

In this case, the action of $\mathbb{R}$ on N is given by

$$\mu(t) = Ad_{\partial_t}(\mathfrak{n}) = e^{t\ ad_{\partial_t}(\mathfrak{n})}. \tag{3.2.10}$$

Another nice feature of almost abelian solvable groups is that a simple criterion exists to determine whether the associated solvmanifold is compact: the group admits a lattice if and only if there exists a $t_0 \neq 0$ for which $\mu(t_0)$ can be conjugated to an integer matrix.

As an example, we can consider two three-dimensional almost abelian solvable algebras

$$\begin{array}{cccccc} \varepsilon_2 & : & [E_2, E_3] = E_1 & \Leftrightarrow & \mathrm{d}e^1 = -e^2 \wedge e^3 \\ & & [E_1, E_3] = -E_2 & \Leftrightarrow & \mathrm{d}e^2 = e^1 \wedge e^3 \end{array} \tag{3.2.11}$$

$$\begin{array}{cccccc} \varepsilon_{1,1} & : & [E_1, E_3] = E_1 & \Leftrightarrow & \mathrm{d}e^1 = -e^1 \wedge e^3 \\ & & [E_2, E_3] = -E_2 & \Leftrightarrow & \mathrm{d}e^2 = e^2 \wedge e^3\,. \end{array} \tag{3.2.12}$$

In the following, we will label the algebras according to their Maurer-Cartan equations. For instance, ε_2 is denoted by $(-23, 13, 0)$, where each entry i gives the result of $\mathrm{d}e^i$.

[3] We denote by $\mathfrak{n}$ the ideal in the algebra and with N the corresponding normal subgroup.

For the algebra $\varepsilon_2 : (-23, 13, 0)$, the nilradical is given by $\mathfrak{n} = \{E_1, E_2\}$ and $\partial_t = E_3$. In this basis, the restriction of the adjoint representation to the nilradical is

$$ad_{\partial_t}(\mathfrak{n}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \tag{3.2.13}$$

which gives a μ matrix of the form

$$\mu(t) = e^{t\ ad_{\partial_t}(\mathfrak{n})} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} . \tag{3.2.14}$$

It is easy to see that, for $t_0 = n\frac{\pi}{2}$, with $n \in \mathbb{Z}^*$, $\mu(t_0)$ is an integer matrix and hence the corresponding manifold is compact.

For the algebra $\varepsilon_{1,1} : (-13, 23, 0)$ the analysis is less straightforward. The nilradical is $\mathfrak{n} = \{E_1, E_2\}$ and again $\partial_t = E_3$. Then, in the (E_1, E_2) basis,

$$ad_{\partial_t}(\mathfrak{n}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \ \mu(t) = e^{t\ ad_{\partial_t}(\mathfrak{n})} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} , \tag{3.2.15}$$

and it is clearly not possible to find a $t_0 \neq 0$ such that $\mu(t_0)$ is an integer. To see whether the group admits a lattice, we then have to go to another basis. In other words, $\mu(t_0)$ will be conjugated to an integer matrix. As in [42], we can define a new basis

$$E_1 \rightarrow \sqrt{\frac{q_2}{q_1}} \frac{E_1 - E_2}{\sqrt{2}} \ , \ E_2 \rightarrow \frac{E_1 + E_2}{\sqrt{2}} \ , \ E_3 \rightarrow \sqrt{q_1 q_2}\, E_3 \ , \tag{3.2.16}$$

with q_1, q_2 strictly positive constants, such that the algebra reads

$$[E_1, E_3] = q_2 E_2 \qquad\qquad [E_2, E_3] = q_1 E_1 \ . \tag{3.2.17}$$

In this new basis

$$ad_{\partial_t}(\mathfrak{n}) = \begin{pmatrix} 0 & -q_1 \\ -q_2 & 0 \end{pmatrix} , \ \mu(t) = \begin{pmatrix} \cosh(\sqrt{q_1 q_2} t) & -\sqrt{\frac{q_1}{q_2}} \sinh(\sqrt{q_1 q_2} t) \\ -\sqrt{\frac{q_2}{q_1}} \sinh(\sqrt{q_1 q_2} t) & \cosh(\sqrt{q_1 q_2} t) \end{pmatrix} , \tag{3.2.18}$$

so that $\mu(t)$ can be made integer with the choice of parameters

$$t_0 \neq 0 \ , \ \cosh(\sqrt{q_1 q_2} t_0) = n_1 \ , \ \frac{q_1}{q_2} = \frac{n_2}{n_3} \ , \ n_2 n_3 = n_1^2 - 1 \ , \ n_{1,2,3} \in \mathbb{Z}^* \ . \tag{3.2.19}$$

Thus also the algebra $\varepsilon_{1,1}$ can be used to construct compact solvmanifolds. Notice that the values $q_1 = q_2 = 1$ are not allowed by the integer condition (3.2.19).

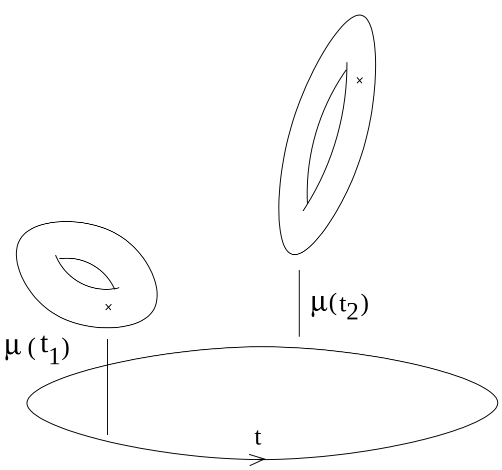


Figure 3.1: Mostow bundle for the solvmanifolds ϵ_2 and $\epsilon_{1,1}$. The base is the circle in the t direction, and due to the nilradical being abelian the fiber is T^2. The fibration is encoded in $\mu(t)$ which is either a rotation or a "hyperbolic rotation" twisting the T^2 when moving along the base.

3.3 Resolution method and known solutions

As discussed in the introduction, nil- and solvmanifolds (twisted tori) are interesting candidates as internal manifolds, to find Minkowski supersymmetric flux backgrounds of type II SUGRA. In particular, all nilmanifolds are proved to be GCY [23], a necessary requirement on the internal manifold. In section 2.4, the constraints to get a supersymmetric solution were rewritten in terms of GCG. One can now look for solutions on these non-CY manifolds, by performing a direct resolution of these constraints, instead of using dualities. Let us give this algorithmic resolution method.

We recall that to obtain a Minkowski SUSY flux background, it is enough to solve the SUSY conditions and the BI of the fluxes: the e.o.m. are then automatically satisfied. Having a supersymmetric background implies both a topological requirement and a differential requirement. The first one is equivalent to the existence of a pair of compatible pure spinors, which gives an $SU(3)\times SU(3)$ structure on E. The second asks this pair to satisfy the SUSY conditions of section 2.4.2. Furthermore, to find a solution the flux must satisfy the BI. For SUSY solutions, the BI are equivalent to the tadpole cancellation conditions. In particular, for a compact internal manifold, the tadpole cancellation requires the presence in the sources of an O-plane. Then the solution must be invariant under the projection defining the O-plane.

For nil- and solvmanifolds, the search for flux vacua is basically algorithmic. The method goes as follow. Given the algebra defining the manifold, one first looks at the O-planes that are compatible with the algebra. Since nil- or solvmanifold are parallelizable manifolds, they admit a basis of real globally defined one-forms e^m, $m = 1\dots 6$, that satisfy the Maurer-Cartan equation (3.2.2). The orientifold projection should leave the set of Maurer-Cartan equations invariant.

For a given O-plane, the first step consists in obtaining general expressions (in terms of the e^m) for the two compatible pure spinors which are also compatible with the orientifold projection. To write compatible pure spinors we will use the properties of $SU(3)$ and $SU(2)$ structures. In particular, once we have written the pure spinors in terms of the forms defining a given structure (2.4.9), the compatibility of the pure spinors is assured by the structure conditions on the forms. The compatibility with the O-plane still has to be satisfied.

The second step consists in solving the SUSY conditions of section 2.4.2

$$(\mathrm{d} - H\wedge)(e^{2A-\phi}\Phi_1) = 0 \ , \tag{3.3.1}$$

$$(\mathrm{d} - H\wedge)(e^{A-\phi}\,\mathrm{Re}(\Phi_2)) = 0 \ , \tag{3.3.2}$$

$$(\mathrm{d} - H\wedge)(e^{3A-\phi}\,\mathrm{Im}(\Phi_2)) = \frac{|a|^2}{8} e^{3A} * \lambda(F) \ . \tag{3.3.3}$$

More precisely, we solve for a closed pure spinor, (3.3.1), and impose the closure of half of the second one (3.3.2). Then we use the third equation, (3.3.3), as a definition of the RR fluxes. Moreover, from the two first equations, one is allowed also to determine the H flux and the dilaton.

To solve the first two equations, one simply has to tune the coefficients in the general expression for the pure spinors. Note that this resolution is greatly simplified if one considers all the coefficients to be constant. This is also means that we work in the large volume limit, where the warp factor is equal to one and the sources are smeared. Even if this approximation can be justified, one can try to reintroduce the warp factor afterwards to localize the solution. Techniques to do so were proposed in [29], and these are successful for solutions with one source. Two intersecting sources are not easy to fully localize. In that case, one can still try partial smearing [43, 44].

In the third step, one asks whether the RR fluxes obtained by equation (3.3.3) can solve the Bianchi identities (2.1.12) with allowed sources. Since (2.4.20) gives the Hodge dual of the fluxes, we need the metric to explicitly use the Hodge star. The metric is defined by the two pure spinors, or equivalently by the structure forms. Out of the holomorphic forms, one deduces the holomorphic directions. Then, the hermitian metric is given in terms of the Kähler form and the almost complex structure

($J_\mu{}^\lambda = i\delta_\mu{}^\lambda,\ J_{\overline{\mu}}{}^{\overline{\lambda}} = -i\delta_{\overline{\mu}}{}^{\overline{\lambda}}$) by

$$g_{\mu\overline{\nu}} = -J_\mu{}^\lambda J_{\lambda\overline{\nu}} \qquad g_{\overline{\mu}\nu} = -J_{\overline{\mu}}{}^{\overline{\lambda}} J_{\overline{\lambda}\nu}\ . \tag{3.3.4}$$

The metric in the e^m basis is obtained by a simple change of basis. Given the metric, one can compute the RR fluxes explicitly. Then we act with the exterior derivative on the RR fluxes, and determine what sources are present. As we consider smeared sources, the BI give directly the directions of the covolumes V^i of the cycles wrapped by the sources. So we write the BI generically as

$$(\mathrm{d} - H\wedge)F = \sum_i Q_i V^i\ , \tag{3.3.5}$$

where Q_i is the charge of the source i. To compute the correct normalisation of the covolumes, we use the fact the covolume can be understood as the Poincaré dual of the form calibrating the source. This yields the following identity (see appendix B.3) [45, 30, 46, 11] in the large volume limit:

$$\left\langle V^i, e^{-\phi}\,\mathrm{Im}(\Phi_2)\right\rangle = \frac{1}{8g_s}\ V\ . \tag{3.3.6}$$

Once we have identified V^i, we read the source charge out of (3.3.5). If it is negative, we conclude we have an orientifold. We should first check that the orientifold initially considered is present. For the other sources, we check whether their directions, read out of the covolumes, are allowed by the calibration conditions. Furthermore, in the case of an unexpected O-plane, we have to verify that the manifold and the solution forms are compatible with its projection. If there appear sources that are not allowed for one of these two reasons, one has to further tune the coefficients if possible, to get rid of them.

Using this method, the authors of [29] found the following solutions on nilmanifolds with non-trivial fluxes.

IIA

M	Algebras	O4	O6	
		t:12	t:30	t:12
n 3.5	$(0,0,0,12,13,23)$		456	
n 5.1	$(0,0,0,0,0,12+34)$	6		
n 5.2	$(0,0,0,0,0,12)$	6		

IIB

	Algebras	O5	
		t:30	t:12
n 3.14	$(0,0,0,12,23,14-35)$	45 + 26	
n 4.4	$(0,0,0,0,12,14+23)$	56	56
n 4.5	$(0,0,0,0,12,34)$	56	56
n 4.6	$(0,0,0,0,12,13)$	56	56
n 4.7	$(0,0,0,0,13+42,14+23)$	56	56
n 5.1	$(0,0,0,0,0,12+34)$	56	56

Table 3.1: Solutions on nilmanifolds, out of [29]

In the first column is given the name of the manifold, associated to the name of the algebra, out of the labelling of [29]. In the second column, the algebra is given in some basis of one-forms. Each entry m of the algebra gives the result of $\mathrm{d}e^m$ (notation of section 3.2). For instance for n 3.14, we have $\mathrm{d}e^1 = \mathrm{d}e^2 = \mathrm{d}e^3 = 0$, $\mathrm{d}e^4 = e^1 \wedge e^2$, etc. In the other columns are indicated the directions wrapped by the O-plane in terms of this basis. The types of the pair of pure spinors are specified (t:ij), giving equivalently the kind of G-structure considered (see section 2.4.1). In particular, note that no intermediate $SU(2)$ structure solution is given, which will be the main subject of the rest of the chapter.

Similarly, few solutions are known on non-nilpotent solvmanifolds [47, 29, 1]:

IIA

M	Algebras	O6	
		t:30	t:12
s 2.5	$(25, -15, r45, -r35, 0, 0)$	136 + 246 146 + 236	136 + 246 146 + 236
$\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}$	$(q_1 25, q_2 15, q_2 45, q_1 35, 0, 0)$	136 + 246 146 + 236	

IIB

	Algebras	O5	
		t:30	t:12
s 2.5	$(25, -15, r45, -r35, 0, 0)$	13 + 24 14 + 23	13 + 24 14 + 23

Table 3.2: Solutions on non-nilpotent solvmanifolds

Among all these solutions, the only ones not T-dual to a warped T^6 configuration are those on n 3.14, s 2.5 and $\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}$. They all have two intersecting sources[4]. We will come back in more details to these solutions.

In the rest of the thesis, we will construct new solutions with non-trivial fluxes, that are not in these tables. First we will present intermediate $SU(2)$ structure solutions on n 3.14 and s 2.5. The analysis of [29] did not take into account the possibility of such a structure in presence of an O-plane. Some examples of intermediate $SU(2)$ structure solutions were found in [11] via T-dualities from a warped T^6 with an $O3$. Here we extend the analysis of [29]: using a similar method, we find new vacua with intermediate $SU(2)$ structure that cannot be T-dualized back to a warped T^6 with an $O3$.

In chapter 4, we present two other new solutions. One is fully localized with one $O6$-plane on s 2.5. To find it, the method described here is modified: the warp factor is introduced back before computing the RR fluxes. The localisation proposal of [29] would not work because the solution is fluxless in the large volume limit. The other solution presented in chapter 4 is found in IIA on a solvmanifold not considered so far. It is given by the algebra $\mathfrak{g}_{5.17}^{p,-p,r} \oplus \mathbb{R}$ which can be understood as the sum

$$\mathfrak{g}_{5.17}^{p,-p,r} \oplus \mathbb{R} \approx s\ 2.5 + p\big(\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}\big)\,, \tag{3.3.7}$$

where p is a real parameter. Therefore, the solution found can be seen as a sum of the two solutions of the previous table. This solution is found by using another method than the one presented here. As we will discuss in the next chapter, this solution is found by using a particular $O(d,d)$ transformation, the twist transformation.

[4]The solutions on n 3.14 and $\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}$ have two O-planes, while those on s 2.5 have one O-plane and one D-brane.

3.4 Intermediate $SU(2)$ structure solutions

The method just described allows to find solutions which cannot be obtained by T-duality from a warped T^6 solution. This is the strategy used in [29], where such solutions were obtained on a nilmanifold n 3.14 and a solvmanifold s 2.5 (see previous tables). On these manifolds, only $SU(3)$ or orthogonal $SU(2)$ structure solutions were looked for, because only those seemed to be compatible with the orientifold projection. Actually, intermediate $SU(2)$ structures[5] can also be compatible with the orientifold projection, when one allows a mixing of the usual $SU(2)$ structure forms [11]. Solutions with such a structure have then been constructed [11], starting from a warped T^6 with an $O3$ and performing some specific T-dualities. Here, we rather apply the method described in the previous section to find non T-dual solutions with intermediate $SU(2)$ structure.

The orientifold projection conditions are not easy to solve for a generic intermediate $SU(2)$ structure. Therefore, we introduce different variables[6] which help to rewrite them in a more tractable way. The SUSY conditions also get simplified in terms of these new variables. This allows us to find new SUSY four-dimensional Minkowski flux vacua of type II string theory with intermediate $SU(2)$ structure. These vacua are not T-dual to a warped T^6 with an $O3$ because the manifolds on which we find them, the same as in [29], do not have the appropriate isometries.

In addition, by going to the limit in which the two internal spinors are parallel or orthogonal, we recover the solutions of [29], and find a new one with $SU(3)$ structure. Finally, we also look at the conditions for an intermediate $SU(2)$ structure solution to be the β-transform of an $SU(3)$ structure solution, and show that it is the case for one of our solutions.

More details on technical aspects of the resolution are given in appendix B.2.

3.4.1 Orientifold projection conditions

When we consider fluxes on compact manifolds, tadpole cancellation requires the inclusion in the solutions of O-plane sources. The presence of O-planes implies that the solution has to be invariant under the action of the orientifold. This imposes some projection conditions on the fields: one has to mod out by $\Omega_{WS}(-1)^{F_L}\sigma$ for $O3/O7$ and $O6$, and by $\Omega_{WS}\sigma$ for $O5/O9$ and $O4/O8$. Ω_{WS} is a worldsheet reflection, F_L is the left-movers fermion number, and σ is an involution on the target space. The orientifold action on the GCG pure spinors were worked-out in [50, 29]. It was concluded in [29] that the orientifold projections are only compatible with $SU(3)$ or orthogonal $SU(2)$ structures. As shown in [11], intermediate $SU(2)$ structures are also compatible with $O5$-, $O6$- and $O7$-planes, if one allows a mixing between the two-forms specifying the structure. Here, we will only consider $O5$- and $O6$-planes.

We first repeat the derivation of the orientifold projection conditions of [11] for $O5$- and $O6$-planes. The resulting conditions on the $SU(2)$ structure forms (j, ω and z) appear to be not very tractable. We then show that it is possible to rewrite these conditions in a more tractable manner, thanks to the introduction of the projection (eigen)basis. We write the pure spinors in these variables, and discuss their relation to the dielectric ones [48, 49]. Finally, we also give the supersymmetry conditions in the projection basis (details on the derivation are in appendix B.2.2), and do the same for some structure conditions in appendix B.2.1.

[5] Intermediate $SU(2)$ structures were presented in section 2.4.1. They correspond to type 01 pure spinors, and so can provide a different geometry.

[6] We call these new variables the projection basis, i.e. the set of structure forms which are "eigenvectors" for the projection. These forms define a new $SU(2)$ structure, obtained by a rotation from the usual one. We show that this $SU(2)$ structure is nothing (modulo a rescaling) but the one appearing with the dielectric pure spinors. These are a rewriting of the GCG pure spinors, used to study the deformations of four-dimensional $\mathcal{N}=4$ Super Yang-Mills in the context of AdS/CFT [48, 49].

The orientifold projection

As shown in [11], the first step to derive the orientifold projection on the pure spinors is to compute those for the internal SUSY parameters. This can be done starting from the projection on the ten-dimensional SUSY spinorial parameters ϵ^i, and then reducing to the internal spinors $\eta^i_\pm$. In our conventions, we get

$$O5 \ : \ \sigma(\eta^1_\pm) = \eta^2_\pm \qquad \sigma(\eta^2_\pm) = \eta^1_\pm \ , \tag{3.4.1}$$
$$O6 \ : \ \sigma(\eta^1_\pm) = \eta^2_\mp \qquad \sigma(\eta^2_\pm) = \eta^1_\mp \ . \tag{3.4.2}$$

σ is the target space reflection in the directions transverse to the O-plane. Using the expressions for the internal spinors given in (2.4.4), we obtain the following projection conditions at the orientifold plane locus:

	$SU(3)$		Intermediate $SU(2)$		Orthogonal $SU(2)$	
	O5	O6	O5	O6	O5	O6
$e^{i\theta_+}$	± 1	free	± 1	free	free	
$e^{i\theta_-}$	free		free	$\mathrm{Re}(e^{i\theta_-} z_\mu)\ \vert\vert$	free	$\mathrm{Re}(e^{i\theta_-} z_\mu)\ \vert\vert$
z	free		$z\ \perp$	$\mathrm{Im}(e^{i\theta_-} z_\mu)\ \perp$	$z\ \perp$	$\mathrm{Im}(e^{i\theta_-} z_\mu)\ \perp$

Table 3.3: Parameters properties at the O-plane locus, according to the source and the G-structure

where μ is a real index, and we assumed that $|a| = |b|$ is invariant under the involution.

As explained in [11], if the G-structures considered are constant, and if we work on nil- or solv-manifolds, these conditions are valid everywhere, not only on the orientifold plane. We will assume that the parameters are indeed constant. Furthermore, θ_- is not directly fixed by the projection. In the following, we will set it to zero: $\theta_- = 0$. Then, the conditions on the one-form z for an $SU(2)$ structure can be reexpressed as

$$\begin{aligned} O5 \ &: \ \sigma(z) = -z \ , \\ O6 \ &: \ \sigma(z) = \overline{z} \ . \end{aligned} \tag{3.4.3}$$

Following [11], starting from the projections on the $\eta^i_\pm$, we derive the projections of the pure spinors $\Phi_\pm$, and from them those for the $SU(2)$ structure forms (using table 3.3). To do this last step, one has to know that, as σ is only the reflection due to the orientifold, it can be distributed on every term of a wedge product. Furthermore, $\lambda(..)$ can also be distributed on wedge products of two forms, provided that one of the two forms is even (see (A.2.10)). So we recover the same projection conditions on the forms as in [11][7]:

$$\begin{aligned} O5 \ : \ & \sigma(j) = (k_{||}^2 - k_\perp^2) j + 2k_{||}k_\perp \,\mathrm{Re}(\omega) \ , \\ & \sigma(\omega) = -k_{||}^2 \omega + k_\perp^2 \overline{\omega} + 2k_{||}k_\perp j \ , \end{aligned} \tag{3.4.4}$$

$$\begin{aligned} O6 \ : \ & \sigma(j) = -(k_{||}^2 - k_\perp^2) j - 2k_{||}k_\perp \,\mathrm{Re}(\omega) \ , \\ & \sigma(\omega) = k_{||}^2 \overline{\omega} - k_\perp^2 \omega - 2k_{||}k_\perp j \ . \end{aligned} \tag{3.4.5}$$

[7] We use slightly different conventions than in [11] but actually one can start with the following general expressions which cover both articles' conventions:

$$\begin{aligned} \Phi_+ &= \frac{|a|^2}{8} e^{i\theta_+} N^2 e^{\frac{z\wedge\overline{z}}{||z||^2}} (k_{||} e^{-ij} - i k_\perp \omega) \ , \\ \Phi_- &= -\frac{|a|^2}{8} e^{i\theta_-} N^2 \sqrt{2} \frac{z}{||z||} \wedge (k_\perp e^{-ij} + i k_{||} \omega) \ , \end{aligned}$$

with $|a|$, $\theta_\pm$, $||z||$, $||\eta_+|| = N$ constant and non-zero, and $k_{||}$, $k_\perp$ constant, and then one gets the same projection conditions.

By introducing as in [11]:

$$O5 \quad : \quad k_{||} = \cos(\varphi),\ k_{\perp} = \sin(\varphi),\ 0 \leq \varphi \leq \frac{\pi}{2}\ , \tag{3.4.6}$$

$$O6 \quad : \quad k_{||} = \cos(\varphi + \frac{\pi}{2}),\ k_{\perp} = \sin(\varphi + \frac{\pi}{2}),\ -\frac{\pi}{2} \leq \varphi \leq 0\ , \tag{3.4.7}$$

we get in both cases the more convenient formulas:

$$\begin{aligned}
\sigma(j) &= \cos(2\varphi) j + \sin(2\varphi)\,\mathrm{Re}(\omega)\ , \\
\sigma(\mathrm{Re}(\omega)) &= \sin(2\varphi) j - \cos(2\varphi)\,\mathrm{Re}(\omega)\ , \\
\sigma(\mathrm{Im}(\omega)) &= -\,\mathrm{Im}(\omega)\ .
\end{aligned} \tag{3.4.8}$$

The projection basis

The projection conditions (3.4.8) are not very tractable. A good idea is to work in the projection (eigen)basis:

$$\begin{aligned}
j_{||} &= \frac{1}{2}(j + \sigma(j))\ , \qquad j_{\perp} = \frac{1}{2}(j - \sigma(j))\ , \\
\mathrm{Re}(\omega)_{||} &= \frac{1}{2}(\mathrm{Re}(\omega) + \sigma(\mathrm{Re}(\omega)))\ , \qquad \mathrm{Re}(\omega)_{\perp} = \frac{1}{2}(\mathrm{Re}(\omega) - \sigma(\mathrm{Re}(\omega)))\ .
\end{aligned} \tag{3.4.9}$$

Using the property $\sigma^2 = 1$ and applying it to the previous equations, we get these more tractable equations:

$$\begin{aligned}
j_{||}\ (1 - \cos(2\varphi)) &= \sin(2\varphi)\ \ \mathrm{Re}(\omega)_{||}\ , \\
j_{\perp}\ (1 + \cos(2\varphi)) &= -\sin(2\varphi)\ \ \mathrm{Re}(\omega)_{\perp}\ .
\end{aligned} \tag{3.4.10}$$

We also get the following equations:

$$\begin{aligned}
j_{||}\ \ \sin(2\varphi) &= (1 + \cos(2\varphi))\ \ \mathrm{Re}(\omega)_{||}\ , \\
j_{\perp}\ \ \sin(2\varphi) &= -(1 - \cos(2\varphi))\ \ \mathrm{Re}(\omega)_{\perp}\ ,
\end{aligned} \tag{3.4.11}$$

which are equivalent to the two equations (3.4.10) if $k_{||}$ and $k_{\perp}$ are non-zero. We now assume it is the case, so we will not use them. Then, for $O6/O5$ (upper/lower), the projection conditions become:

$$\begin{aligned}
\sigma(\mathrm{Re}(z)) &= \pm\ \mathrm{Re}(z)\ , \\
\sigma(\mathrm{Im}(z)) &= -\,\mathrm{Im}(z)\ , \\
\sigma(\mathrm{Im}(\omega)) &= -\,\mathrm{Im}(\omega)\ , \\
j_{||} &= \mp \left(\frac{k_{\perp}}{k_{||}}\right)^{\pm 1}\ \mathrm{Re}(\omega)_{||}\ , \\
j_{\perp} &= \pm \left(\frac{k_{\perp}}{k_{||}}\right)^{\mp 1}\ \mathrm{Re}(\omega)_{\perp}\ .
\end{aligned} \tag{3.4.12}$$

In this form, the projection conditions are now much more tractable.

Pure spinors and dielectric variables

We now rewrite the pure spinors in terms of the variables of the projection basis. Let us first give these relations (they are nothing but a rewriting of the two last projection conditions given in (3.4.12)):

$$\begin{aligned}
\mathrm{IIA} \quad &: \quad k_{||} j_{||} + k_{\perp}\,\mathrm{Re}(\omega)_{||} = 0\ , \qquad -k_{\perp} j_{\perp} + k_{||}\,\mathrm{Re}(\omega)_{\perp} = 0\ , \\
\mathrm{IIB} \quad &: \quad -k_{\perp} j_{||} + k_{||}\,\mathrm{Re}(\omega)_{||} = 0\ , \qquad k_{||} j_{\perp} + k_{\perp}\,\mathrm{Re}(\omega)_{\perp} = 0\ .
\end{aligned} \tag{3.4.13}$$

These allow to write the following relations valid for both theories:

$$
\begin{aligned}
-\sin(\varphi)j + \cos(\varphi)\,\mathrm{Re}(\omega) &= \frac{1}{\cos(\varphi)}\,\mathrm{Re}(\omega)_{\perp} = -\frac{1}{\sin(\varphi)} j_{\perp}\ , \\
\cos(\varphi)j + \sin(\varphi)\,\mathrm{Re}(\omega) &= \frac{1}{\sin(\varphi)}\,\mathrm{Re}(\omega)_{||} = \frac{1}{\cos(\varphi)} j_{||}\ .
\end{aligned}
\tag{3.4.14}
$$

These last relations (3.4.14) can also be found by using the definitions of $j_{||}$, $j_{\perp}$, $\mathrm{Re}(\omega)_{||}$, and $\mathrm{Re}(\omega)_{\perp}$. One can notice in the previous relation a rotation. We will come back to it soon.

We can now rewrite the pure spinors in (2.4.10) using the projection basis and the relations (3.4.14):

$$
\begin{aligned}
\mathrm{IIA}\quad :\quad \Phi_{+} &= \frac{|a|^2}{8} e^{i\theta_+} k_{||}\ e^{\frac{1}{2} z\wedge\overline{z} - \frac{i}{k_{||}k_{\perp}}\mathrm{Re}(\omega)_{\perp} + \frac{k_{\perp}}{k_{||}}\mathrm{Im}(\omega)}\ , \\
\Phi_{-} &= -\frac{|a|^2}{8} k_{\perp}\ z\wedge e^{\frac{i}{k_{||}k_{\perp}}\mathrm{Re}(\omega)_{||} - \frac{k_{||}}{k_{\perp}}\mathrm{Im}(\omega)}\ ,
\end{aligned}
\tag{3.4.15}
$$

$$
\begin{aligned}
\mathrm{IIB}\quad :\quad \Phi_{+} &= \frac{|a|^2}{8} e^{i\theta_+} k_{||}\ e^{\frac{1}{2} z\wedge\overline{z} - \frac{i}{k_{||}k_{\perp}}\mathrm{Re}(\omega)_{||} + \frac{k_{\perp}}{k_{||}}\mathrm{Im}(\omega)}\ , \\
\Phi_{-} &= -\frac{|a|^2}{8} k_{\perp}\ z\wedge e^{\frac{i}{k_{||}k_{\perp}}\mathrm{Re}(\omega)_{\perp} - \frac{k_{||}}{k_{\perp}}\mathrm{Im}(\omega)}\ .
\end{aligned}
\tag{3.4.16}
$$

Recently, an alternative parametrization of the internal supersymmetry parameters, and consequently of the pure spinors, was given in [48] and further discussed in [49]

$$
\begin{aligned}
\eta^1_+ &= a\left(\cos(\Psi)\eta_{+D} - \sin(\Psi)\frac{z\eta_{-D}}{2}\right)\ , \\
\eta^2_+ &= a e^{-i\theta_+}\left(\cos(\Psi)\eta_{+D} + \sin(\Psi)\frac{z\eta_{-D}}{2}\right)\ ,
\end{aligned}
\tag{3.4.17}
$$

where we still have θ_+ as the difference of phase between η^1_+ and η^2_+, a and z are the same as before, $||\eta_{+D}|| = 1$ and Ψ is an angle such as $0 \leq \Psi \leq \frac{\pi}{4}$. This different choice was proposed in order to study deformations of four-dimensional $\mathcal{N} = 4$ Super Yang-Mills in the context of AdS/CFT. Typically those deformations should describe the near horizon geometry of some sort of dielectric branes, hence the name dielectric for the spinor η_{+D}. Note that η_{+D} is nothing but (once the phases of the two spinors are equalled) the mean spinor between η^1_+ and η^2_+, i.e. somehow their bisector: $\eta_{+D} = \frac{1}{2a\ \cos(\Psi)}(\eta^1_+ + e^{i\theta_+}\eta^2_+)$. We have the corresponding picture:

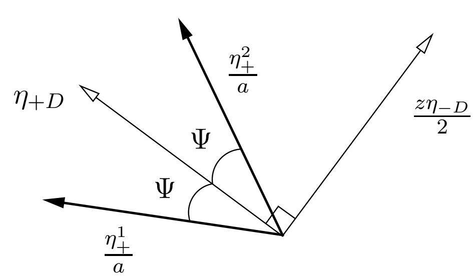


Figure 3.2: The different spinors and angles (with $\theta_+ = 0$)

One can relate the dielectric ansatz to the previous one, (2.4.4), with

$$
k_{||} = \cos(\varphi) = \cos(2\Psi),\ \ k_{\perp} = \sin(\varphi) = \sin(2\Psi)\ , \tag{3.4.18}
$$

$$
\eta_{+D} = \cos(\frac{\varphi}{2})\eta_{+} + \sin(\frac{\varphi}{2})\frac{z\eta_{-}}{2}\ . \tag{3.4.19}
$$

Working with η_{+D} and $\frac{z\eta_{-D}}{2}$ instead of η_+ and $\frac{z\eta_-}{2}$ means working with a new $SU(2)$ structure. The latter is obtained by a rotation from the previous one, as one can also get by computing the relations between the $SU(2)$ structure two-forms:

$$\begin{aligned} j_D &= k_{||} j + k_\perp \,\mathrm{Re}(\omega) \ , \\ \mathrm{Re}(\omega_D) &= -k_\perp j + k_{||}\,\mathrm{Re}(\omega) \ , \\ \mathrm{Im}(\omega_D) &= \mathrm{Im}(\omega) \ . \end{aligned} \tag{3.4.20}$$

The pure spinors obtained from (3.4.17) [48, 49] are the dielectric pure spinors[8]

$$\begin{aligned} \Phi_+ &= \frac{|a|^2}{8} e^{i\theta_+} k_{||}\ e^{\frac{1}{2} z\wedge \overline{z} - \frac{i}{k_{||}} j_D + \frac{k_\perp}{k_{||}} \mathrm{Im}(\omega_D)} \ , \\ \Phi_- &= -\frac{|a|^2}{8} k_\perp\ z\wedge e^{\frac{i}{k_\perp}\mathrm{Re}(\omega_D) - \frac{k_{||}}{k_\perp}\mathrm{Im}(\omega_D)} \ . \end{aligned} \tag{3.4.21}$$

Comparing the definitions of the two-forms (3.4.14) and (3.4.20), or the expressions for the pure spinors, (3.4.15), (3.4.16) and (3.4.21), we get (for IIA/IIB)

$$\begin{aligned} j_D &= \frac{1}{k_\perp}\mathrm{Re}(\omega)_{\perp/||} \ , \\ \mathrm{Re}(\omega_D) &= \frac{1}{k_{||}}\mathrm{Re}(\omega)_{||/\perp} \ , \\ \mathrm{Im}(\omega_D) &= \mathrm{Im}(\omega) \ . \end{aligned} \tag{3.4.22}$$

Thus the dielectric $SU(2)$ structure variables are nothing but the eigenbasis of the orientifold projection (modulo a rescaling) ! Actually, this can be easily understood from the transformation properties of η_{+D} under the orientifold projection[9]

$$\begin{aligned} O5 &:\ \sigma(\eta_{\pm D}) = e^{i\theta_+}\eta_{\pm D} \ , \\ O6 &:\ \sigma(\eta_{\pm D}) = \eta_{\mp D} \ . \end{aligned} \tag{3.4.23}$$

Then the $SU(2)$ bilinears constructed from it will get at most a phase and a conjugation when being applied σ, hence the three real two-forms j_D, $\mathrm{Re}(\omega_D)$ and $\mathrm{Im}(\omega_D)$ are in the projection eigenbasis, as given by (3.4.22).

Beside providing a tractable basis to solve the orientifold projection conditions, the dielectric variables/projection basis lead to simpler expressions of the pure spinors and so much simpler SUSY conditions (see further). So this $SU(2)$ structure is a better choice to solve our problem, and we will express the equations to be solved in terms of these variables. We rewrite the SUSY conditions in terms of the projection basis, and in appendix B.2.1, we rewrite similarly a set of $SU(2)$ structure conditions (implying the compatibility conditions, see appendix A.2.2).

3.4.2 SUSY equations in the projection basis

In appendix B.2.2, the SUSY equations (2.4.18), (2.4.19), and (2.4.20), are expanded in terms of forms for general expressions of the pure spinors (2.4.9), with $\theta_- = 0$. The set of equations can be simplified by using the $SU(2)$ structure conditions. We consider more simplifications by choosing $|a|^2 = e^A$, and going to the large volume limit, i.e. $A = 0$ and $e^\phi = g_s$ constant. This is indeed the regime in which we will look for solutions. The freedom in θ_+ is not fixed, except in IIB where we use the O5 projection: $e^{i\theta_+} = \pm 1$. Moreover, we choose to look only for intermediate $SU(2)$ structure

[8] The computation is the same as using (2.4.10) and introducing the dielectric $SU(2)$ structure variables via (3.4.20).
[9] To get them, we recall that we have $e^{i\theta_+} = \pm 1$ for an $O5$, and one has to use (3.4.1) and (3.4.2).

solutions, i.e. with $k_{||}$ and $k_\perp$ constant and non-zero. Here is the result in terms of the projection basis variables.

$$
\begin{aligned}
\text{IIA} \quad : \quad & g_s * F_4 = -k_\perp \mathrm{d}(\mathrm{Im}(z)) \\
& k_{||} H = k_\perp \mathrm{d}(\mathrm{Im}(\omega)) \\
& g_s * F_2 = -k_{||}\mathrm{d}(\mathrm{Im}(z)) \wedge \mathrm{Im}(\omega) + \frac{1}{k_{||}}\mathrm{d}(\mathrm{Re}(\omega)_{||}) \wedge \mathrm{Re}(z) - \frac{1}{k_\perp} H \wedge \mathrm{Im}(z) \\
& g_s * F_0 = \frac{1}{2} k_\perp \mathrm{d}(\mathrm{Im}(z)) \wedge \mathrm{Im}(\omega)^2 + \frac{1}{k_{||}} H \wedge \mathrm{Re}(z) \wedge \mathrm{Re}(\omega)_{||}
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{d}(\mathrm{Re}(z)) = 0 \\
& \mathrm{d}(\mathrm{Re}(\omega)_\perp) = k_{||} k_\perp \,\mathrm{Re}(z) \wedge \mathrm{d}(\mathrm{Im}(z)) \\
& H \wedge \mathrm{Re}(z) = -\frac{k_\perp}{k_{||}} \mathrm{d}(\mathrm{Im}(z) \wedge \mathrm{Re}(\omega)_{||}) \ ,
\end{aligned}
\tag{3.4.24}
$$

$$
\begin{aligned}
\text{IIB} \quad : \quad & k_{||} H = k_\perp \mathrm{d}(\mathrm{Im}(\omega)) \\
& k_\perp e^{-i\theta_+} g_s * F_3 = \mathrm{d}(\mathrm{Re}(\omega)_{||}) \\
& k_\perp e^{-i\theta_+} g_s * F_1 = H \wedge \mathrm{Re}(\omega)_{||}
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{d}(\mathrm{Re}(z)) = 0 \\
& \mathrm{d}(\mathrm{Im}(z)) = 0 \\
& \mathrm{Re}(z) \wedge H = -\frac{k_\perp}{k_{||}} \mathrm{Im}(z) \wedge \mathrm{d}(\mathrm{Re}(\omega)_\perp) \\
& \mathrm{Im}(z) \wedge H = \frac{k_\perp}{k_{||}} \mathrm{Re}(z) \wedge \mathrm{d}(\mathrm{Re}(\omega)_\perp) \\
& \mathrm{Re}(z) \wedge \mathrm{Im}(z) \wedge \mathrm{d}(\mathrm{Re}(\omega)_{||}) = -H \wedge \mathrm{Im}(\omega) \ .
\end{aligned}
\tag{3.4.25}
$$

3.4.3 Intermediate $SU(2)$ structure solutions

In order to find intermediate $SU(2)$ structure solutions, we use the method described in section 3.3. As we look for intermediate $SU(2)$ structure, we take $k_{||}k_\perp \neq 0$ and constant (we have $k_\perp = \sqrt{1-k_{||}^2}$). As discussed previously, the other coefficients in the solutions are also taken to be constant. In particular, we choose $|a|^2 = e^A$, and go to the large volume limit, i.e. where $A = 0$ and $e^\phi = g_s$ is constant.

To solve the SUSY conditions, we use the $SU(2)$ structure defined by the projection basis. The forms are determined by imposing the orientifold projection conditions given by (3.4.12) and the $SU(2)$ structure conditions (B.2.1) and (B.2.2). Then the pure spinors in the projection basis (3.4.15) or (3.4.16) are compatible (see appendix A.2.2), and we impose the SUSY equations given by (3.4.24) or (3.4.25).

The next step of the procedure is to compute the RR fields and, therefore the metric. This is done in the initial basis for the $SU(2)$ structure forms, which defines the local complex one-forms $(z^1, z^2, z, \overline{z}^1, \overline{z}^2, \overline{z})$. We identify one of them with the holomorphic one-form z of the $SU(2)$ structure, and write the real and the holomorphic two-forms of the $SU(2)$ structure as

$$
\omega = z^1 \wedge z^2 \qquad j = \frac{i}{2}(t_1 z^1 \wedge \overline{z}^1 + t_2 z^2 \wedge \overline{z}^2 + b z^1 \wedge \overline{z}^2 - \overline{b}\overline{z}^1 \wedge z^2) \ ,
\tag{3.4.26}
$$

with $b = b_r + ib_i$ and t_1, t_2, b_r, b_i real[10]. We will give our solutions in the previous form[11]. As discussed in section 3.3, we can define trivially an almost complex structure in this basis, and compute from it and the Kähler form (2.2.19) the hermitian metric. In this local complex basis, we obtain generically[12]

$$g = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & t_1 & b & 0 \\ 0 & 0 & 0 & \overline{b} & t_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ t_1 & \overline{b} & 0 & 0 & 0 & 0 \\ b & t_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.4.27}$$

Its definite-positiveness is given by $g_{\mu\overline{\mu}} > 0$ for any μ. It is equivalent to $t_1 > 0$ and $t_2 > 0$. Consider now the volume form. This can be generically written as

$$V = \sqrt{|g|}\; e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6 \ , \tag{3.4.28}$$

where we chose the orientability convention $\epsilon_{123456} = 1$ (see appendix A.1). Comparing this expression from the volume for obtained from the Mukai pairing of the pure spinors (see below equation (2.4.13)) sets other constraints on the signs of the parameters of the solution. Note that here, using (2.4.9) for the pure spinors, and then the form (3.4.26) of the solutions, we get for the volume form

$$V = \frac{1}{8i} z \wedge \overline{z} \wedge \omega \wedge \overline{\omega} = -\,\mathrm{Re}(z^1) \wedge \mathrm{Im}(z^1) \wedge \mathrm{Re}(z^2) \wedge \mathrm{Im}(z^2) \wedge \mathrm{Re}(z) \wedge \mathrm{Im}(z) \ . \tag{3.4.29}$$

The last step of the procedure, once the RR fluxes are computed, is to get the BI (3.3.5). The identification of the covolumes of the sources is given by the formula (3.3.6), which can be rewritten here using (2.4.9) as

$$\begin{aligned} \mathrm{IIA} \ &: \ V^i \wedge \Big(\mathrm{Re}(z) \wedge \mathrm{Re}(\omega)_{||} - k_{||}^2 \ \mathrm{Im}(z) \wedge \mathrm{Im}(\omega)\Big) = k_{||}\ V \ , \\ \mathrm{IIB} \ &: \ V^i \wedge \Big(\mathrm{Re}(\omega)_{||} + k_{\perp} k_{||} \ \mathrm{Re}(z) \wedge \mathrm{Im}(z)\Big) = k_{\perp} e^{-i\theta_+}\ V \ . \end{aligned} \tag{3.4.30}$$

Our search, on nil- or solvmanifolds, for solutions with intermediate $SU(2)$ structure is not meant to be exhaustive. Our interest is to verify the possibility of having solutions of this kind that are not obtainable via T-duality. We look at the manifolds for which non T-dual solutions with $SU(3)$ or orthogonal $SU(2)$ structure were found in [29]: the nilmanifold n 3.14 of algebra $(0,0,0,12,23,14-35)$, and the solvmanifold s 2.5 of algebra $(25,-15,r45,-r35,0,0)$. We had the intuition that some intermediate $SU(2)$ structure might be found on them, which might give back solutions of [29] in the limits $k_{\perp/||} \to 0$. We indeed find three new solutions, and in section 3.4.4, we discuss their limits to the solutions of [29].

On the manifolds considered, the compatible directions for an $O5$ or an $O6$ are given by

Manifold	$O5$	$O6$
n 3.14	13, 15, 26, 34, 45	none
s 2.5	13, 14, 23, 24, 56	125, 136, 146, 236, 246, 345

Table 3.4: Directions of the possible $O5$ and $O6$ on the manifolds considered

[10] Note that the choice of this basis is not unique. In particular, this freedom will appear when taking the limits, see section 3.4.4.

[11] Note that the metric we will then compute from it will be block diagonal, so the remaining $SU(2)$ structure conditions, namely the contractions with z and $\overline{z}$, are clearly satisfied by these expressions.

[12] Note that we give here the coefficients of the metric tensor: they are symmetric, but do not have to be real, since only the tensor has to be real. In practice, one needs the metric in the real basis (e^m, $m = 1..6$). To get it, one has first to perform a change of basis.

Among these possibilities, we look for solutions only for one set of directions on each manifold.

Let us now give the solutions found. For more general parametrisation of these solutions, look at the corresponding paper [1].

First solution

We look for a type IIB solution on the nilmanifold n 3.14. We start with only one $O5$ in the 45 directions, but it will turn out to be a second one, along 26. The solution is first given by

$$\begin{aligned} z &= \tau_0(e^1 - ie^3) \ , \\ \mathrm{Re}(\omega)_{||} &= \frac{k_\perp^2 r_1 r_2}{r_3} e^4 \wedge e^5 - r_3 e^2 \wedge e^6 \ , \\ \mathrm{Re}(\omega)_\perp &= k_{||}^2 r_1 e^2 \wedge e^4 - r_2 e^5 \wedge e^6 \ , \\ \mathrm{Im}(\omega) &= r_1 e^2 \wedge e^5 + r_2 e^4 \wedge e^6 \ . \end{aligned} \tag{3.4.31}$$

where $r_3 > 0$, $r_1 r_2 > 0$ are real parameters, and τ_0 is a non-zero complex one. The solution is clearly compatible with the projections of both sources (under the involutions σ_{45} and σ_{26}). After finding the solutions, we expressed them as in (3.4.26)

$$\begin{aligned} & z = \tau_0(e^1 - ie^3) \ , \ z^1 = -r_1(k_{||}^2 e^4 + ie^5) + r_3 e^6 \ , \ z^2 = e^2 - \frac{r_2}{r_3} i(e^4 + ie^5) \ , \\ & b_r = 0 \ , \ b_i = -\frac{k_{||}}{k_\perp} \ , \ t_1 = \frac{r_2}{k_{||} k_\perp r_1 r_3} \ , \ t_2 = \frac{k_{||} r_1 r_3}{k_\perp r_2} \ . \end{aligned} \tag{3.4.32}$$

The metric is diagonal in the e^i basis, its coefficients are given by[13]

$$g = \mathrm{diag} \left(|\tau_0|^2, \frac{k_{||} r_1 r_3}{k_\perp r_2}, |\tau_0|^2, \frac{k_{||} k_\perp r_1 r_2}{r_3}, \frac{k_\perp r_1 r_2}{k_{||} r_3}, \frac{r_2 r_3}{k_\perp k_{||} r_1} \right) \ , \tag{3.4.33}$$

and out of the volume form we get

$$\sqrt{|g|} = |\tau_0|^2 r_1 r_2 \ . \tag{3.4.34}$$

Then, we get the following fluxes:

$$\begin{aligned} H &= \frac{k_\perp r_2}{k_{||}} \left(-e^3 \wedge e^4 \wedge e^5 + e^1 \wedge e^2 \wedge e^6 \right) \ , \\ g_s F_3 &= \frac{e^{i\theta_+} r_2}{r_1} \left(\frac{k_\perp r_1 r_2}{r_3} \ (e^3 \wedge e^4 \wedge e^6 + \frac{1}{k_{||}^2} e^1 \wedge e^5 \wedge e^6) - \frac{r_3}{k_\perp} \ (-\frac{1}{k_{||}^2} e^3 \wedge e^5 \wedge e^6 + e^1 \wedge e^4 \wedge e^6) \right) \ , \\ g_s F_1 &= \frac{e^{i\theta_+}}{k_{||} r_1} \left(r_3 e^1 - \frac{k_\perp^2 r_1 r_2}{r_3} e^3 \right) \ . \end{aligned} \tag{3.4.35}$$

We then compute the Bianchi identities:

$$\begin{aligned} & \mathrm{d}(F_1) = 0 \ , \ H \wedge F_3 = 0 \ , \\ & g_s \mathrm{d}(F_3) - H \wedge g_s F_1 = -\frac{2 r_2}{k_{||}^2 r_1 \sqrt{|g|}} \left(\frac{(k_\perp r_1 r_2)^2}{r_3^2} \ V^1 + \frac{r_3^2}{k_\perp^2} \ V^2 \right) \ . \end{aligned} \tag{3.4.36}$$

[13]Note that our convention $||z||^2 = \overline{z}^\mu z_\mu = 2$ is already implemented in the metric, by its construction from the Kähler form in which this norm appears. One can verify this point by computing this norm using either the hermitian or the real basis metric. Then, $|\tau_0|^2$ has nothing to do with this norm, but is only the measure related to the metric coefficients, in the real basis.

The only sources are those of F_3. There are two of them, one along the directions 45 and the other along 26, of covolumes given by

$$V^1 = \frac{k_\perp e^{-i\theta_+}|\tau_0|^2 r_3}{k_\perp^2}\ e^1\wedge e^2\wedge e^3\wedge e^6\ ,\ V^2 = \frac{k_\perp e^{-i\theta_+}|\tau_0|^2 r_1 r_2}{r_3}\ e^1\wedge e^3\wedge e^4\wedge e^5\ . \tag{3.4.37}$$

One can read directly the charges (see (3.3.5)) and see that $Q_1 < 0$, $Q_2 < 0$, hence we have two O-plane sources. Both are compatible with the manifold and the solution.

Note that we will not find any T-dual solution to this first solution, while the two next solutions are T-duals to one another. This can be understood from table 3.4 since no $O6$ is compatible with n 3.14.

Second solution

We look for IIB solutions on the solvmanifold s 2.5. Its algebra admits a parameter $r \in \mathbb{Z}$. We consider an $O5$ in the 13 directions. Once again we will get a second source along 24. Our solution is given by

$$\begin{aligned} z &= r r_5 e^5 + i r_6 e^6\ , \\ \mathrm{Re}(\omega)_{||} &= \frac{k_\perp^2 r r_1^2}{r_3} e^1\wedge e^3 + r_3 e^2\wedge e^4\ , \\ \mathrm{Re}(\omega)_\perp &= r_2 e^1\wedge e^2 - \frac{k_{||}^2 r r_1^2}{r_2} e^3\wedge e^4\ , \\ \mathrm{Im}(\omega) &= r_1(e^1\wedge e^4 - r e^2\wedge e^3)\ , \end{aligned} \tag{3.4.38}$$

where r_1, r_2, r_3, r_5, r_6 are real parameters satisfying $r_1 r_2 r_3 > 0$, and $r_5 r_6 > 0$. Furthermore, we must have $r^2 = 1$. This solution is clearly compatible with the projections under σ_{13} and σ_{24}. The solution is then expressed with the following z^i:

$$\begin{aligned} &z = r r_5 e^5 + i r_6 e^6\ ,\ z^1 = r_2 e^1 + i r r_1 e^3 - r_3 e^4\ ,\ z^2 = e^2 + \frac{k_\perp^2 r r_1^2}{r_2 r_3} e^3 + i\frac{r_1}{r_2} e^4\ , \\ &b_r = 0\ ,\ b_i = \frac{k_\perp}{k_{||}}\ ,\ t_1 = \frac{k_\perp r_1}{k_{||} r_2 r_3}\ ,\ t_2 = \frac{r_2 r_3}{k_\perp k_{||} r_1}\ . \end{aligned} \tag{3.4.39}$$

The metric is then

$$g = \mathrm{diag}\ \left(\frac{k_\perp r_1 r_2}{k_{||} r_3}, \frac{r_2 r_3}{k_\perp k_{||} r_1}, \frac{k_\perp k_{||} r_1^3}{r_2 r_3}, \frac{k_{||} r_1 r_3}{k_\perp r_2}, r_5^2, r_6^2\right)\ ,\ \sqrt{|g|} = r_5 r_6 r_1^2\ . \tag{3.4.40}$$

The fluxes are:

$$\begin{aligned} &H = 0\ ,\ F_1 = 0\ , \\ &g_s F_3 = \frac{e^{i\theta_+}(-k_\perp^2 r_1^2 + r_3^2)\ r_6}{k_\perp r_3 r_5}(e^2\wedge e^3\wedge e^6 + r\ e^1\wedge e^4\wedge e^6)\ , \end{aligned} \tag{3.4.41}$$

and the non-trivial BI

$$g_s \mathrm{d}(F_3) = -\frac{2(r_3^2 - k_\perp^2 r_1^2)\ r_6 r_1^2}{r_5\sqrt{|g|}}\left(\frac{1}{k_\perp^2 r_1^2} V^1 - \frac{1}{r_3^2} V^2\right)\ . \tag{3.4.42}$$

We see that F_3 has two sources, the one along 13 as expected, and we discover a second one along 24. As before, we compute the covolumes and get

$$V^1 = -\frac{k_\perp e^{-i\theta_+} r_5 r_6 r_1^2}{r_3}\ e^1\wedge e^3\wedge e^5\wedge e^6\ ,\ V^2 = -\frac{e^{-i\theta_+} r r_5 r_6 r_3}{k_\perp}\ e^2\wedge e^4\wedge e^5\wedge e^6\ . \tag{3.4.43}$$

The nature of the sources depends on the sign of their charges, which depends here on the value of the parameters. But we can clearly see that there is one O-plane and one D-brane. In both cases, the O-plane is compatible with the manifold. Note also that we clearly have $\sum_i Q_i < 0$.

Third solution

We look for IIA solutions on the solvmanifold s 2.5, but now with an $O6$ in the 136 directions. We will get a second source along 246. We are going to see that this solution is T-dual to the second one, so there will be a lot of similarities between the two. The solution is given by:

$$\begin{aligned} z &= -irr_5 e^5 + r_6 e^6 \ , \\ \mathrm{Re}(\omega)_{||} &= -\frac{k_{||}^2 r r_1^2}{r_3} e^1 \wedge e^3 - r_3 e^2 \wedge e^4 \ , \\ \mathrm{Re}(\omega)_{\perp} &= -r_2 e^1 \wedge e^2 + \frac{k_{\perp}^2 r r_1^2}{r_2} e^3 \wedge e^4 \ , \\ \mathrm{Im}(\omega) &= -r_1 (e^1 \wedge e^4 - r e^2 \wedge e^3) \ , \end{aligned} \tag{3.4.44}$$

where r_1, r_2, r_3, r_5, r_6 are real parameters satisfying $r_1 r_2 r_3 > 0$ and $r_5 r_6 > 0$, and r has to be ± 1. The solution is clearly compatible with the projections under σ_{136} and σ_{246}. The general solution is then expressed with the following z^i:

$$\begin{aligned} & z = -irr_5 e^5 + r_6 e^6 \ , \ z^1 = -r_2 e^1 - irr_1 e^3 + r_3 e^4 \ , \ z^2 = e^2 + \frac{k_{||}^2 r r_1^2}{r_2 r_3} e^3 + i \frac{r_1}{r_2} e^4 \ , \\ & b_r = 0 \ , \ b_i = -\frac{k_{||}}{k_{\perp}} \ , \ t_1 = \frac{k_{||} r_1}{k_{\perp} r_2 r_3} \ , \ t_2 = \frac{r_2 r_3}{k_{\perp} k_{||} r_1} \ . \end{aligned} \tag{3.4.45}$$

The metric is then:

$$g = \mathrm{diag}\ \left(\frac{k_{||} r_1 r_2}{k_{\perp} r_3}, \frac{r_2 r_3}{k_{||} k_{\perp} r_1}, \frac{k_{||} k_{\perp} r_1^3}{r_2 r_3}, \frac{k_{\perp} r_1 r_3}{k_{||} r_2}, r_5^2, r_6^2 \right) \ , \ \sqrt{|g|} = r_5 r_6 r_1^2 \ . \tag{3.4.46}$$

The fluxes are:

$$\begin{aligned} & H = 0 \ , \ F_0 = 0 \ , \ F_4 = 0 \ , \\ & g_s F_2 = -\frac{(-k_{||}^2 r_1^2 + r_3^2)}{k_{||} r_3 r_5} (e^2 \wedge e^3 + r\ e^1 \wedge e^4) \ , \end{aligned} \tag{3.4.47}$$

and the only non-trivial BI is

$$g_s \mathrm{d}(F_2) = -\frac{2(r_3^2 - k_{||}^2 r_1^2)\ r_6 r_1^2}{r_5 \sqrt{|g|}} \left(\frac{1}{k_{||}^2 r_1^2} V^1 - \frac{1}{r_3^2} V^2\right) \ . \tag{3.4.48}$$

We see that F_2 has two sources, the one along 136 as expected, and we discover a second one along 246. As before, we compute the covolumes and get

$$V^1 = \frac{k_{||} r_5 r_1^2}{r_3}\ e^1 \wedge e^3 \wedge e^5, \ V^2 = \frac{r r_5 r_3}{k_{||}}\ e^2 \wedge e^4 \wedge e^5 \ . \tag{3.4.49}$$

The nature of the sources depends on the sign of their charges, which depends here on the value of the parameters. But we can clearly see that there is one O-plane and one D-brane. In both cases, the O-plane is compatible with the manifold. Note also that we clearly have $\sum_i Q_i < 0$.

Let us now show that this solution is T-dual to the previous one, by a T-duality along the e^6 direction as can be seen from the sources. As discussed in section 2.3.3, in order to perform T-duality on the second solution pure spinors, we should act on the normalized pure spinors (2.4.13) with the operator $T = e^6 \wedge + \iota_6$. As there is no B-field involved here, it means

$$e^{-\phi_T} \Phi_T = e^{-\phi}\ T \cdot \Phi \ , \tag{3.4.50}$$

where Φ_T is the T-dual pure spinor, and ϕ_T the T-dual dilaton. Let us apply this operator on the second solution pure spinors. The T-duality is applied along one direction, so we go from IIB to IIA, and $\Phi_\pm$ of the second and third solutions should roughly be exchanged. The T-duality direction is a component of z, hence according to the $SU(2)$ structure contraction properties (A.2.3) and (A.2.6), it easy to see that the pure spinors (3.4.15) and (3.4.16) will indeed be exchanged, provided

$$\mathrm{Re}(\omega)_{||T} = -\mathrm{Re}(\omega)_{||} \ , \ \mathrm{Re}(\omega)_{\perp T} = -\mathrm{Re}(\omega)_{\perp} \ , \ \mathrm{Im}(\omega)_T = -\mathrm{Im}(\omega) \ , \ k_{||T} = k_{\perp} \ . \qquad (3.4.51)$$

This is clearly verified when comparing the second and third solution forms (3.4.38) and (3.4.44). Furthermore, the z and dilaton are transformed as

$$z_T = -e^{i\theta_+} i(rr_5 e^5 - i\frac{1}{r_6}e^6) \ , \ e^{-\phi_T} = \frac{r_6}{g_s} \ , \qquad (3.4.52)$$

where $g_s, r, r_5, r_6, \theta_+$ are parameters of the starting solution. The transformation of the dilaton is standard, and is due to the inversion of the radius in the 6-direction. The g_s of the third solution is then slightly redefined with respect to that of the second solution. About z, we recall that $e^{i\theta_+} = \pm 1$ since we start from an O5. The T-dual phase is given by $e^{i\theta_{+T}} = -i$, and it is free, because not fixed by the O6. Furthermore, to be precise, the z_T can be identified with $\overline{z}$, the conjugate of z of the third solution. In the end, we do get the third solution (same fields), provided we take a slightly different $SU(3)$ structure: $J = j - \frac{i}{2} z \wedge \overline{z}$, $\Omega = \overline{z} \wedge \omega$.

Note that the exchange of $k_{||}$ and $k_\perp$ for such a T-duality explains why an $SU(3)$ structure is then dual to a orthogonal $SU(2)$ structure, as it is the case for the solutions in [29], and as we will see now, taking the limit of our solutions.

3.4.4 $SU(3)$ or orthogonal $SU(2)$ structures limits

In [29], $SU(3)$ or orthogonal $SU(2)$ structure solutions were found on the manifolds we have just studied. So it is interesting to see what happens to our solutions when we take one of those two limits: it would be somehow natural to recover the solutions of [29]. It was at first the kind of intuition that led us to look for intermediate $SU(2)$ solutions on these manifolds. To take the limit on our solutions, one has two options: taking the limit of the pure spinors, or taking the limit of the structure forms. Taking the limit of the pure spinors might not be a good idea. Indeed, we know pure spinors have different types for each G-structure, so there might be a problem when taking the limit[14]. More precisely, only one of the two spinors keeps the same type in the limit, so this pure spinor might transform smoothly, while the other might not. This is summed-up in this table:

$$\begin{array}{cccccc} & SU(3) & & \text{Int. } SU(2) & & \text{Orth. } SU(2) \\ \Phi_+ & 0 & \longleftarrow & 0 & \dashrightarrow & 2 \\ \Phi_- & 3 & \dashleftarrow & 1 & \longrightarrow & 1 \end{array} \qquad (3.4.53)$$

with the numbers indicating the types of the pure spinors, the plain arrows indicating the smooth limits and the dashed ones indicating the limits where there might be a problem. We recover this point when considering the dielectric pure spinors expressions (3.4.21): when one replaces first in (3.4.21) j_D and $\mathrm{Re}(\omega_D)$ by their expressions (3.4.20), and then takes the limit, one does not get the correct expressions for the limit pure spinors. To get them right, one has to use the following prescription: first take the limit of j_D and $\mathrm{Re}(\omega_D)$, and then the limit of the corresponding expression for (3.4.21).

This prescription is more in favour of the second option: taking the limit of the structure forms, and that is what we will do. Looking at the expressions of the dielectric forms j_D and ω_D in (3.4.20),

[14] In particular, the type of the closed pure spinor determines the geometry (see section 2.4.1), so if this type changes when taking the limit, the geometry gets modified, which does seem not very smooth.

we see that their limits give straightforwardly the forms of the limit structures. Actually, we prefer to use the projection basis $\mathrm{Re}(\omega)_{||}$, $\mathrm{Re}(\omega)_{\perp}$ and $\mathrm{Im}(\omega)$, as we gave our solutions with these variables. More precisely, we are going to take the limit of $\mathrm{Im}(\omega)$ and $\frac{1}{k_{..}}\mathrm{Re}(\omega)_{..}$, where .. stands for $||$ or $\perp$. Doing so, we also recover the forms of the limit structures, as one can see from (3.4.22) or (3.4.14). We get[15]

	$SU(3)$ $k_{\perp} \to 0$	Orthogonal $SU(2)$ $k_{||} \to 0$
IIA	$\frac{1}{k_{||}}\mathrm{Re}(\omega)_{||} \to \mathrm{Re}(\omega)$ $\frac{1}{k_{\perp}}\mathrm{Re}(\omega)_{\perp} \to j$	$\frac{1}{k_{\perp}}\mathrm{Re}(\omega)_{\perp} \to \mathrm{Re}(\omega)$ $-\frac{1}{k_{||}}\mathrm{Re}(\omega)_{||} \to j$
IIB	$\frac{1}{k_{||}}\mathrm{Re}(\omega)_{\perp} \to \mathrm{Re}(\omega)$ $\frac{1}{k_{\perp}}\mathrm{Re}(\omega)_{||} \to j$	$\frac{1}{k_{\perp}}\mathrm{Re}(\omega)_{||} \to \mathrm{Re}(\omega)$ $-\frac{1}{k_{||}}\mathrm{Re}(\omega)_{\perp} \to j$

It is clear that $\frac{1}{k_{..}}\mathrm{Re}(\omega)_{..}$ is not the best choice for taking the limit since $k_{||}$ or $k_{\perp}$, assumed non-zero, have to go to zero[16]. Indeed, one can see from the previous arrays that ω is always recovered smoothly while j is not recovered very easily. For instance in the case IIA and $SU(3)$ limit, $k_{\perp}$ and $\mathrm{Re}(\omega)_{\perp}$ both go to zero, and only their ratio is supposed to give back j. To get a well-defined limit, we should have a non-zero j, so we must have in this example $\mathrm{Re}(\omega)_{\perp} \sim k_{\perp} f_2 \to 0$, where f_2 stands for a constant real two-form. Imposing this last condition will give us the behaviour of some of our parameters. It can also sometimes lead to inconsistencies such as the volume form going to zero, and then we can say that there is no limit solution.

By first studying the limit to j, we then get conditions on the behaviour of our parameters: some go to zero in a specific way. Using them, we work out the limit to ω (extrapolated to Ω_3 in the $SU(3)$ case), and manage to get the z^i of [29] solutions, noted z_s^i, by factorizing the form as they do. Then we work out completely the limit to j (extrapolated to J in the $SU(3)$ case), and find the corresponding t_{is} and b_s as in (3.4.26) to get their solution. Finally, we verify that we have the same fluxes as they do when taking the limit on ours.

The validity of this procedure could be discussed further. In particular, we do recover the structure forms found in [29] (modulo global normalisation factors) as we find maps between their parameters and ours. But there is a possible mismatch for the H flux in the orthogonal $SU(2)$ limit, as one can see from its definition in the SUSY conditions (3.4.24) or (3.4.25). Indeed, if we did not find any H in the intermediate case, we cannot take its limit to recover an H in the orthogonal $SU(2)$ limit, while the SUSY conditions allow for a non-trivial H in this limit. This situation will happen for our third solution, as they do find a possible H in [29] while we do not. For our second solution, this problem could also have occurred, but no H was found in [29]. Note that if there is a mismatch with H, then there is possibly one with the other fluxes, as they can be defined out of H.

Limits of the first solution

Let us first consider the $SU(3)$ limit of the first solution which should correspond to "Model 1" of [29] (same theory, same manifold, same orientifold(s)). Imposing that $\mathrm{Re}(\omega)_{||}$ goes to zero ($\sim k_{\perp}$) and comparing with their J_s gives this behaviour for our parameter: $r_3 \sim k_{\perp}$ (with a possible positive constant that we will not consider for simplicity). Note that a priori in our solution r_3 could not be zero, so can we put it to zero in this limit? One criterion to verify that the limit is well-defined is that the six-form volume must not go to zero. And r_3 actually does not appear in it, as one can see from the determinant of the metric, so it is fine (furthermore none of the metric coefficients goes to

[15] Note that we recover in these limits the fact that j, $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$ of the limit structures are the projection eigenbasis.

[16] The difficulties that can occur are related to the one just explained for the pure spinors, since they both are related to the assumption of $k_{||}$ and $k_{\perp}$ being non-zero.

zero). So using this behaviour of our parameter and the limits given in the array, we get their Ω_{3s} and J_s with a global normalisation difference. The normalisation factor affects both Ω_3 and J so that the normalisation condition (2.2.14) is still satisfied. One just have to rescale some of the z_s^i and the t_{is} to match the one we have when taking the limit. We get[17]

$$\begin{aligned} & z^1 = \tau_0(e^1 - ie^3),\ z^2 = e^2 + i\tau e^6,\ z^3 = r_1(e^4 + ie^5),\ \text{with } \tau = -\frac{r_2}{r_1}\ , \\ & t_1 = 1,\ t_2 = -\frac{1}{\tau},\ t_3 = -\tau\ . \end{aligned} \tag{3.4.54}$$

Looking at our fluxes, we get that $H \to 0$ as in [29], and we deduce that $F_1 \to 0$ from the SUSY conditions (3.4.25). Moreover, taking the limit on our F_3, we recover the solution of [29], once the t_{is} are rescaled.

Let us now consider the $SU(2)$ limit. Looking at the condition $\mathrm{Re}(\omega)_\perp \to 0$, one gets at least $r_2 \to 0$. But this is not allowed, because the volume form would go to zero (see the determinant of the metric). Note that allowing for r_1 to diverge to maintain a finite volume is not appropriate, since $\mathrm{Im}(\omega)$ would then diverge. So we recover the statement of [29]: there is no orthogonal $SU(2)$ limit. Note that a T-dual on this manifold to the $SU(3)$ limit would have been a orthogonal $SU(2)$ structure with an $O6$. Then, the fact that there is no orthogonal $SU(2)$ on this manifold can also be understood by the fact that there is no $O6$ compatible, according to table 3.4.

Limits of the second solution

For more general parametrisations of the limit structure forms, look at the corresponding paper [1].

Let us first consider the $SU(3)$ limit of the second solution. We mention first that no corresponding solution is mentioned in [29]. There can be several reasons for this, among them one can be that there is no solution with fluxes which is non T-dual to a warped T^6 with an $O3$. We actually do find such a solution, which should be the T-dual to the orthogonal $SU(2)$ limit of our third solution (see further). So we will use similar notations. Considering as usual $\mathrm{Re}(\omega)_{||} \to 0$ ($\sim k_\perp$), we get $r_3 \sim xk_\perp$ with x a real constant. As for the previous solution, $r_3 \to 0$ is not allowed for an intermediate $SU(2)$ structure. With the same arguments as before, it can actually be allowed in the $SU(3)$ limit (see the determinant of the metric and the metric components). Using this behaviour, we get:

$$\Omega_{SU(3)} = r_2\ (rr_5\ e^5 + ir_6\ e^6) \wedge (e^1 - \tau e^3) \wedge (e^2 - r\tau e^4)\ , \tag{3.4.55}$$

with $\tau = -i\frac{rr_1}{r_2}$ (clearly of the same form as the orthogonal $SU(2)$ limit of the third solution).

Let us now consider the fluxes. We get $H = 0$ and then $F_1 = 0$. In the simpler case chosen for the parameters, we get a non-trivial F_3 in the $SU(3)$ limit:

$$g_s F_{3\ SU(3)} = \frac{e^{i\theta_+}(-r_1^2 + x^2)\ r_6}{xr_5}(e^2 \wedge e^3 \wedge e^6 + r\ e^1 \wedge e^4 \wedge e^6)\ . \tag{3.4.56}$$

The solution obtained in the limit is compatible with the two sources appearing when computing the BI.

Let us now consider the orthogonal $SU(2)$ limit of the second solution, which should correspond to "Model 2" in [29] (taking $r = 1$). Our z is clearly the same as theirs. By imposing that $\mathrm{Re}(\omega)_\perp$ goes to zero ($\sim k_{||}$) and comparing its limit with their j_s, we get this behaviour for one of our parameters:

[17] Note that we have here an example of a different choice for the z^i, mentioned in footnote 10. The way we recovered their solution is then a reparametrization: we computed the two-form in the limit and then refactorized it in the way they did.

$r_2 \sim -xk_{||}$ where x is a real constant. It was forbidden in our solution to put this parameter to zero but when one looks at the metric components and its determinant, one sees it can be allowed in the orthogonal $SU(2)$ limit. The solution given in [29] is the following:

$$\omega_s = (e^1 + i(-\tau_2^2 e^2 + \tau_2^1 e^4 + \tau_3^1 e^5)) \wedge (e^3 + i(\tau_2^2 e^4 + \tau_3^2 e^5 + (2\frac{b_s}{t_{2s}}\tau_2^2 + \frac{t_{1s}}{t_{2s}}\tau_2^1)e^2)) \ , \tag{3.4.57}$$

with all parameters real, and $t_{2s} = \frac{1+b_s^2}{t_{1s}}$. When taking the limit on our forms, we get the same result, with a global normalisation factor difference: our ω $_{\text{orthogonal } SU(2)}$ and our j $_{\text{orthogonal } SU(2)}$ are obtained by multiplying theirs by $l = \frac{r_1^2}{r_3}$. Apart from this normalisation, we manage to recover their solution with[18]

$$\tau_2^1 = \tau_3^1 = \tau_3^2 = 0 \ , \ \tau_2^2 = \frac{rr_3}{r_1} \ , \ t_{1s} = -\frac{x}{rr_1}, \ b_s = 0 \ . \tag{3.4.58}$$

We recover both their j_s and their ω_s with a factor l difference, so that the normalisation condition (2.2.16) stays correct for us and for them. As this normalisation condition involves l^2, we have the choice on the sign of the factor in j (we took $+l$), which is related to the sign of r_3. It is then related to the sign of the t_i appearing.

Let us now look at the fluxes. We have only an F_3 as they do. By taking the limit of our $d(F_3)$, we exactly get theirs, multiplied by l as it should be.

Limits of the third solution

For more general parametrisations of the limit structure forms, look at the corresponding paper [1].

We already mentioned that this solution was the T-dual of the second one. In [29], they also mention this point for the limit structures: their "Model 3", which should be the $SU(3)$ limit of our solution (with $r = 1$), is mentioned to be the T-dual of their "Model 2", which is the orthogonal $SU(2)$ limit of our second solution. So this $SU(3)$ limit of our solution must match their "Model 3", and we do not have to consider further the $SU(3)$ limit. Note for instance that we get the "same" (T-dual) limit behaviour of our parameter: $r_2 \sim -xk_\perp$ where x is a real constant.

Let us now consider the orthogonal $SU(2)$ limit of our third solution, which corresponds to the "Model 4" in [29]. With the same reasoning, it is probably the T-dual to the $SU(3)$ limit of our second solution, that did not match to any solution found in [29]. We first note that our z matches theirs, modulo a global i factor. This difference is due to a different phase convention for the $O6$. Let us look at the other forms. As usual, considering the limit of $\mathrm{Re}(\omega)_{||}$ and comparing it to j_s imposes $r_3 \sim xk_{||}$ with x a real constant. Once again, r_3 going to zero can be allowed in this limit (see the determinant of the metric). Using this behaviour, we get the solution of [29] by taking the limit on our forms. In [29] they have:

$$\omega_s = (\tau_1^1 e^1 + \tau_2^1 e^3) \wedge (\tau_1^2 e^2 + \frac{\tau_2^1}{\tau_1^1}\tau_1^2 e^4 + \tau_3^2 e^5) \ , \tag{3.4.59}$$

with complex parameters, and we match it and j_s with:

$$\begin{aligned} &\tau_3^2 = 0 \ , \ \tau_1^1 = -\frac{r_2}{\tau_1^2} \ , \ \tau_2^1 = -ir\frac{r_1}{\tau_1^2} \ , \\ &t_{1s} = r\frac{r_1|\tau_1^2|^2}{r_2 x} = \frac{1}{t_{2s}} \ , \end{aligned} \tag{3.4.60}$$

[18] Note that the complicated expressions for the parameters are related to the freedom left in choosing different expressions for the z^i (we did not take the same as them) as mentioned in footnote 10, and our t_i are different for the same reason. Note also that we manage to recover this general expression with non-zero coefficients, with the more general parametrisation of structure forms given in [1].

where τ_1^2 is not fixed. Note $t_{1s}t_{2s} = 1$ is here the normalisation condition (2.2.16).

Let us now look at the fluxes. There are slight differences, due to H and z. As explained previously, we do not get any H, while they do: this is an artefact of our procedure. Note nevertheless that their H is more constrained than it appears to be in [29], once one imposes it to be real. By taking the limit on our $d(F_2)$, we get exactly theirs, modulo the factor coming from z (related to the difference between our z and theirs), and the following map we have to impose: $|\tau_1^2|^4 = r_1^2$. This last condition can seem surprising, but this difference is probably related to the absence of H in our limit. Besides, note that the solution obtained is clearly compatible with the sources appearing.

3.5 β-transform

In [51], a dynamical $SU(2)$ structure solution is presented on a non-compact manifold. This solution is obtained by a local β-transform (see section 2.3 for a definition) of an $SU(3)$ structure solution. Since a dynamical $SU(2)$ structure solution is at most points an intermediate $SU(2)$ structure solution, one can ask whether the solutions found previously are as well obtainable by β-transforms, in particular from their $SU(3)$ limit that we discussed. We will first discuss the conditions under which this would be possible, provided some assumptions. Then we will give one example where these assumptions and conditions are verified: the intermediate $SU(2)$ structure solution found previously on n 3.14.

We use the $SU(3)$ pure spinors given in (2.4.11), with a particular $SU(3)$ structure given by $J_l = j_l + \frac{i}{2} z_l \wedge \overline{z}_l$, and $\Omega_l = z_l \wedge \omega_l$. For the intermediate $SU(2)$ structure, we use the dielectric pure spinors given in (3.4.21). For both, we take $\theta_\pm = 0$, where the value for θ_+ can be understood from the O5 projection. We also work in the large volume limit, i.e. $A = 0$ and $|a|^2 = 1$.

One of the SUSY equations ((2.4.18) in IIA, (2.4.19) in IIB), with previous values taken for the parameters, fixes the value for the dilaton (see appendix B.2.2 about these different integration constants)

$$e^{\phi} = g_s k_{||} \text{ for int. SU(2)} \ , \quad e^{\phi} = g_s \text{ for SU(3)} \ . \tag{3.5.1}$$

Given one solution with $SU(3)$ structure, and another one with an intermediate $SU(2)$ structure, a β transformation relating the two means

$$e^{\beta \llcorner} \Psi_{\pm\ SU(3)} = \Psi_{\pm\ \text{Interm}\ SU(2)} \ . \tag{3.5.2}$$

where we use the normalised pure spinors $\Psi_\pm$ (2.4.13) of E to act with an $O(d,d)$ transformation like the β-transform. Here, these spinors take the following form for an intermediate $SU(2)$ structure

$$\begin{aligned} \Psi_+ &= \frac{1}{g_s} e^{-B + \frac{1}{2} z \wedge \overline{z} - \frac{i}{k_{||}} j_D + \frac{k_\perp}{k_{||}} \mathrm{Im}(\omega_D)} \\ \Psi_- &= -\frac{k_\perp}{g_s k_{||}} z \wedge e^{-B + \frac{i}{k_\perp} \mathrm{Re}(\omega_D) - \frac{k_{||}}{k_\perp} \mathrm{Im}(\omega_D)} \ , \end{aligned} \tag{3.5.3}$$

and for an $SU(3)$ structure

$$\begin{aligned} \Psi_+ &= \frac{1}{g_s} e^{-i(j_l + \frac{i}{2} z_l \wedge \overline{z}_l)} \\ \Psi_- &= -\frac{i}{g_s} z_l \wedge \omega_l \ , \end{aligned} \tag{3.5.4}$$

where we take $B_{SU(3)} = 0$ (particular case).

Provided some assumptions we are now going to give, the equation (3.5.2) can be rewritten as a simpler set of conditions.

3.5.1 Assumptions and conditions

In order to derive the conditions equivalent to (3.5.2), we are going to make the following (simplifying) assumptions. Note that only the first one is satisfied in [51], even if (3.5.2) is satisfied there.

1. We take β to be real, $\beta^2 = 0$, and β to be orthogonal to z_l (meaning it does not act on z_l).
2. We already chose $||z|| = ||z_l||$ $(= \sqrt{2})$ since we took the previous formulas for the pure spinors. It is not the case in [51] where there is a "$\sqrt{h}$" difference.
3. We take j_l, ω_l and j_D, ω_D (and the new B-field) to span the same four-dimensional space, orthogonal to the one-forms z_l, z. It is not the case in [51] because of some non-trivial fibration (connection terms modify the one-forms, as discussed in [2]). This assumption enables us to give a simple relation between ω_l and ω_D, B. Furthermore, the $SU(2)$ structure conditions (B.2.3) are then more easily satisfied.
4. We take $i\ \beta \llcorner \omega_l$ to be real. There is no such phase in [51].

Given these assumptions, one develops (3.5.2) and gets the following set of conditions:

$$\begin{aligned}
& i\beta\llcorner\omega_l = \frac{k_\perp}{k_{||}}\ ,\ z = z_l\ ,\\
& j_D = k_{||} j_l\ ,\ \mathrm{Re}(\omega_D) = k_{||}\,\mathrm{Re}(\omega_l)\ ,\ \mathrm{Im}(\omega_D) = k_{||}^2\,\mathrm{Im}(\omega_l) - \frac{k_{||}k_\perp}{2}\beta\llcorner j_l^2\ ,\\
& B = \frac{k_\perp}{k_{||}}\,\mathrm{Im}(\omega_D) + \frac{1}{2}\beta\llcorner j_l^2 = k_{||}k_\perp\,\mathrm{Im}(\omega_l) + \frac{k_{||}^2}{2}\beta\llcorner j_l^2\ ,\\
& \beta\llcorner j_l = 0\ ,\ \left(\beta\llcorner j_l^2\right)^2 = 0\ ,\ j_l\wedge(\beta\llcorner j_l^2) = 0\ .
\end{aligned} \tag{3.5.5}$$

Let us note that the SUSY conditions for intermediate $SU(2)$ structures give $H = \mathrm{d}\left(\frac{k_\perp}{k_{||}}\,\mathrm{Im}(\omega_D)\right)$, so we have a solution if $\beta\llcorner j_l^2$ is closed.

As a simple example, let us consider the following solution on T^6:

$$\begin{aligned}
& e^i = dx^i\ ,\ dz^1 = e^1 + ie^2\ ,\ dz^2 = e^3 + ie^4\ ,\ dz^3 = e^5 + ie^6\ ,\\
& z_l = dz^3\ ,\ j_l = \frac{i}{2}(dz^1\wedge d\overline{z}^1 + dz^2\wedge d\overline{z}^2) = e^1\wedge e^2 + e^3\wedge e^4\ ,\\
& \omega_l = dz^1\wedge dz^2 = e^1\wedge e^3 - e^2\wedge e^4 + i(e^1\wedge e^4 + e^2\wedge e^3)\ .
\end{aligned} \tag{3.5.6}$$

Choosing

$$\beta = \frac{k_\perp}{k_{||}}\iota_{\partial_1}\llcorner\iota_{\partial_4}\llcorner\ , \tag{3.5.7}$$

all assumptions and conditions are satisfied with the following dielectric forms:

$$\begin{aligned}
& z = e^5 + ie^6\ ,\ j_D = k_{||}(e^1\wedge e^2 + e^3\wedge e^4)\ ,\ B = k_{||}k_\perp e^1\wedge e^4\ ,\\
& \mathrm{Re}(\omega_D) = k_{||}(e^1\wedge e^3 - e^2\wedge e^4)\ ,\ \mathrm{Im}(\omega_D) = k_{||}^2 e^1\wedge e^4 + e^2\wedge e^3\ .
\end{aligned} \tag{3.5.8}$$

This set should be an intermediate $SU(2)$ solution, since this β is an element of the T-duality group. This example is interesting to see how things work, and in particular, that the condition

$$j_D^2 = \mathrm{Re}(\omega_D)^2 = \mathrm{Im}(\omega_D)^2 \neq 0 \tag{3.5.9}$$

can be satisfied. This is not obvious from the generic formulas (3.5.5). Note as well that we get $H = 0$.

3.5.2 Solutions on n 3.14

Let us now check that the $SU(3)$ solution found in [29], and the intermediate $SU(2)$ solution of [1] presented previously, can be mapped via a β-transform. Considering a simple region of the moduli space (i.e. taking some simple values for some solution parameters), it was possible in section 3.4.4 to recover solutions of [29] by taking the $SU(3)$ limit of the intermediate $SU(2)$ solutions. So we will consider the $SU(3)$ structure solution via the limit obtained previously, i.e.

$$\begin{aligned} &z_l = \tau_0(e^1 - ie^3)\ ,\ j_l = -r_1^2\tau e^4 \wedge e^5 - e^2 \wedge e^6\ , \\ &\mathrm{Re}(\omega_l) = r_1(e^2 \wedge e^4 + \tau e^5 \wedge e^6)\ ,\ \mathrm{Im}(\omega_l) = r_1(e^2 \wedge e^5 - \tau e^4 \wedge e^6)\ , \end{aligned} \tag{3.5.10}$$

where $\tau = -\frac{r_2}{r_1}$. The intermediate $SU(2)$ structure solution is given in (3.4.31). We rewrite it thanks to (3.4.22) as

$$\begin{aligned} &z = \tau_0(e^1 - ie^3)\ ,\ j_D = -\frac{k_\perp r_1^2 \tau}{r_3} e^4 \wedge e^5 - \frac{r_3}{k_\perp} e^2 \wedge e^6 \\ &\mathrm{Re}(\omega_D) = r_1(k_{||} e^2 \wedge e^4 + \frac{\tau}{k_{||}} e^5 \wedge e^6)\ ,\ \mathrm{Im}(\omega_D) = r_1(e^2 \wedge e^5 - \tau e^4 \wedge e^6) \end{aligned} \tag{3.5.11}$$

where $\tau = -\frac{r_2}{r_1}$. In the intermediate $SU(2)$ structure solution, a priori one cannot distinguish the constants r_1, r_2, r_3 from $k_{||}$ and $k_\perp$ which are just other constants. But when taking the $SU(3)$ limit, it is important to know if there is any relation between them. In particular, it was realised that one has to choose $r_3 \sim k_\perp$ to get correct limits. The dependence in $k_\perp$ cannot be missed because it goes to 0. We did not try so far to work out the dependence in $k_{||}$, which is more subtle since it goes to 1. Though we can always redefine the parameters with some dependence in $k_{||}$, this will not change the intermediate $SU(2)$ solution, and will give the same limit. Therefore, let us redefine

$$r_3 = k_\perp k_{||}\ ,\ r_2 = k_{||}^2 \tilde{r}_2\ ,\ \tilde{\tau} = -\frac{\tilde{r}_2}{r_1}\ . \tag{3.5.12}$$

Now the two solutions are rewritten as

$$\begin{aligned} &z = \tau_0(e^1 - ie^3)\ ,\ j_D = -k_{||}(r_1^2 \tilde{\tau} e^4 \wedge e^5 + e^2 \wedge e^6)\ , \\ &\mathrm{Re}(\omega_D) = k_{||} r_1(e^2 \wedge e^4 + \tilde{\tau} e^5 \wedge e^6)\ ,\ \mathrm{Im}(\omega_D) = r_1(e^2 \wedge e^5 - k_{||}^2 \tilde{\tau} e^4 \wedge e^6)\ , \\ \\ &z_l = \tau_0(e^1 - ie^3)\ ,\ j_l = -r_1^2 \tilde{\tau} e^4 \wedge e^5 - e^2 \wedge e^6\ , \\ &\mathrm{Re}(\omega_l) = r_1(e^2 \wedge e^4 + \tilde{\tau} e^5 \wedge e^6)\ ,\ \mathrm{Im}(\omega_l) = r_1(e^2 \wedge e^5 - \tilde{\tau} e^4 \wedge e^6)\ . \end{aligned} \tag{3.5.13}$$

Now we can satisfy the conditions for the β-transform. In particular we have

$$z = z_l\ ,\ j_D = k_{||} j_l\ ,\ \mathrm{Re}(\omega_D) = k_{||}\,\mathrm{Re}(\omega_l)\ . \tag{3.5.14}$$

Furthermore, choosing

$$\beta = \frac{k_\perp}{k_{||} r_1 \tilde{\tau}} \iota_6 \llcorner \iota_4 \llcorner\ , \tag{3.5.15}$$

all conditions (3.5.5) and the assumptions are satisfied. In addition, we get the correct H-field because $\beta \llcorner j_l^2 \propto e^2 \wedge e^5$ is closed on this manifold. So the two solutions are indeed mapped by a β-transform.

Chapter 4

Twist transformation in type II and heterotic string

4.1 Introduction

In the previous chapter, we discussed explicit examples of Minkowski supersymmetric flux backgrounds on solvmanifolds. In section 3.3, we gave a list of known solutions.

The solutions found on non-nilpotent solvmanifolds are not T-duals to a warped T^6 configuration. Among the solutions on nilmanifolds, only those corresponding to the algebra n 3.14 are also not T-duals to a warped T^6. Notice also that, in type IIB, starting from a T^6 with an $O3$-plane and a non-trivial B-field, and performing two (independent) T-dualities, one ends-up in the same theory on a nilmanifold with the first Betti number being equal to 5 or 4 [33, 52]. On the contrary, n 3.14 has its first Betti number $b_1(M) = 3$. One can ask whether all these solutions, T-duals or not, could be related by some more general transformation. We present in this chapter such a transformation, that we call the twist.

The idea is to construct a $GL(d)$ operator that maps the basis of one-forms of the torus into the Maurer-Cartan one-forms of a given solvmanifold. The Maurer-Cartan one-forms reflect the topology of the manifold. Therefore, such an operator should encode the topology of the solvmanifold reached by the transformation. In a way, the matrix $\mu(t)$ of almost abelian solvmanifolds (see section 3.2) is already an example of a $GL(d)$ operator encoding the topology of the manifold. It encodes the fibration of the Mostow bundle. Our operator will actually be very close to the $\mu(t)$ matrices of such solvmanifolds.

If one is able to relate forms on the torus to forms on the solvmanifolds, it is tempting to try to relate whole solutions. In the GCG formalism, a natural way to transform a solution is to act on its pure spinors with an $O(d,d)$ element (see section 2.3.3). A well-known example is the action of the T-duality group $O(n,n)$ (see [53] for a review) on a manifold with n isometries. Therefore, it is natural to embed the topology changing $GL(d)$ transformation into an $O(d,d)$ transformation, acting equivalently on forms and on vectors of the generalized tangent bundle. Given this embedding, we can extend the transformation with other ingredients. These will transform the B-field with a B-transform, the metric with some scalings, and the dilaton will be shifted accordingly. We can additionally allow for a pair of $U(1)$ actions given by the phases discussed in (2.3.44). This way, we will be able to transform the whole NSNS sector as desired. The $O(d,d)$ action being done on the generalized tangent bundle, the RR fluxes are not transformed directly, but via the pure spinors and the SUSY equation defining them[1].

Thus we propose to use the twist transformation to relate solutions on the torus to solutions on

[1] The general transformation we deduce for the RR fields mixes NSNS and RR sectors, which is not the case of T-duality. One way to do this kind of mixing is via U-duality, but our transformation does not seem related to it.

solvmanifolds. Differently from T-duality, the twist is a local $O(d,d)$ transformation. Then, while the former is a symmetry of the equations of motion, the twist in general is not. Nevertheless, we can work out general constraints for it to preserve the SUSY conditions. In some cases, these constraints are simple enough to be solved. Then, one can use the transformation as a solution generating technique. For instance, we are able to relate all type IIB solutions on nilmanifolds presented in section 3.3, including the seemingly isolated non T-dual solution on n 3.14. For solvmanifolds, we can as well recover the type IIB non T-dual solution on s 2.5. We also use the twist to construct a new solution on a new solvmanifold. We discuss as well the possibility to get non-geometric solutions.

Twist transformations can also be applied in the heterotic string to connect two supersymmetric flux solutions recently discovered on manifolds with different topologies[2]. To so, we discuss how to formulate the heterotic SUSY conditions in terms of GCG.

Before entering the details, let us mention the content of the related appendix C. We first give a more detailed construction of one-forms of solvmanifolds, and give a list of solvmanifolds in terms of these globally-defined one-forms. Then we discuss the possible non-geometric T-duals of solvmanifold solutions. Finally, in the heterotic context, we extend the generalized tangent bundle to include the gauge bundle, in order to transform the gauge fields directly via a subset of local $O(d+16,d+16)$ transformations.

4.2 Twist construction of globally defined one-forms on a solvmanifold

In this section we show how to construct the globally defined one-forms of a given six-dimensional solvmanifold[3] from those of a torus T^6. We name this construction the twist. To start with, we do not bother about global issues related to the compactness of the manifold and we focus on the one-forms of the corresponding solvable group. Given such a group G, we want to relate one-forms on $T^*\mathbb{R}^6$ to those of $T^*G = \mathfrak{g}^*$

$$A \begin{pmatrix} \mathrm{d}x^1 \\ \vdots \\ \mathrm{d}x^6 \end{pmatrix} = \begin{pmatrix} e^1 \\ \vdots \\ e^6 \end{pmatrix} . \tag{4.2.1}$$

A is a local matrix that should contain the bundle structure of G. Indeed, the one-forms $e^{m=1\dots6}$ constructed from A should verify the Maurer-Cartan equation (3.2.2) which describes the topology of G. In addition, these forms should be globally defined, but we will come back later to this question.

In appendix C.1.1, we show how the construction works in general for almost nilpotent and nilpotent groups. Here we focus on two simple cases: a nilpotent group with only one fibration, and an almost abelian solvable group, defined in section 3.2.

- <u>Nilpotent group with only one fibration</u>:

  For nilpotent groups, there is no particular difficulty with compactness: one can always find a lattice that takes the group into a nilmanifold. So with some abuse of language, we rather talk here of a nilmanifold. We consider the case where the nilmanifold is simply one fibration of $\mathcal{F} = T^{k_{\mathcal{F}}}$ over the base $\mathcal{B} = T^{k_{\mathcal{B}}}$. In that case, we take[4]

$$A = e^{-\frac{1}{2} ad_{\mathcal{B}}(\mathfrak{g})} = e^{-\frac{1}{2}\sum_{l\in\mathcal{B}} x^l ad_{E_l}(\mathfrak{g})} , \tag{4.2.2}$$

[2]These solutions have been related before by the so-called Kähler/non-Kähler transition: the relation is established via a complicated and indirect chain of dualities involving a lift to M-theory [54, 55, 56, 57, 58, 59, 60, 61, 62].

[3]We refer to section 3.2 for a general discussion on solvmanifolds.

[4]This formula does correspond to a subcase of the one (C.1.9) given in appendix C.1.1 for the general case. Indeed, for only one fibration, $ad_{\mathcal{B}}(X)$ is zero if $X \in \mathcal{F}$ as we can see from the Lie bracket definition. Furthermore, $ad_{\mathcal{B}}(X) \in \mathcal{F}$ for $X \in \mathcal{B}$ since there is only one fibration. So the two formulas for A_N match.

where the x^l are coefficients parametrizing generic elements of $\mathcal{B}$, and that we can consider as coordinates, i.e. $E_l = \partial_{x^l}$. The forms constructed this way verify the Maurer-Cartan equation (see appendix C.1.1).

Let us give an example: the Heisenberg algebra considered in section 3.2. As we can see from the Maurer-Cartan equation, the base and fiber are given by the following basis: $\mathcal{B} = \{E_2, E_3\}$, $\mathcal{F} = \{E_1\}$. A generic element E of $\mathcal{B}$ is then parametrized as: $E = x^2 E_2 + x^3 E_3$. So the action is given by[5]

$$ad_{\mathcal{B}}(\mathfrak{g}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -x^3 & x^2 & 0 \end{pmatrix} \text{ in the basis } (E_2, E_3, E_1) \ \Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}x^3 & -\frac{1}{2}x^2 & 1 \end{pmatrix} .$$

Out of this A we construct the one-forms, and we can verify that the Maurer-Cartan equation is satisfied:

$$\mathrm{d}e^1 = -e^2 \wedge e^3 .$$

From (4.2.2) it follows that the generic form of A for one fibration is always given by

$$A = \left(\begin{array}{c|c} 1_{k_{\mathcal{B}}} & 0 \\ \hline \mathcal{A} & 1_{k_{\mathcal{F}}} \end{array} \right) , \tag{4.2.3}$$

where the $\mathcal{A}$ block gives the connection of the fibration. For instance for the Heisenberg algebra, the connection one-form is given by $\frac{1}{2}(x^3 \mathrm{d}x^2 - x^2 \mathrm{d}x^3)$. We will come back to (4.2.3) and give further examples.

- Almost abelian solvable group:

From the Mostow bundle (3.2.6), it is natural to identify x^6 with the coordinate t parametrising the $\mathbb{R}$ subalgebra and to take the corresponding one-form as $\mathrm{d}x^6 = \mathrm{d}t$. Then the matrix A takes the form

$$A = \begin{pmatrix} A_M & 0 \\ 0 & 1 \end{pmatrix} , \tag{4.2.4}$$

where A_M is a five-dimensional matrix given by

$$A_M = \mu(-t) = \mu(t)^{-1} = e^{-t \ ad_{\partial_t}(\mathfrak{n})} . \tag{4.2.5}$$

It is not surprising to use the operator $\mu(t)$: as discussed in section 3.2, this operator provides the structure of the Mostow bundle, and the criterion for the compactness of the manifolds. These two properties are those we expect to find in the A map (4.2.1). It is straightforward to show that the forms constructed this way verify the Maurer-Cartan equation (see (C.1.7)):

$$\mathrm{d}e^i = \mathrm{d}(e^{-t \ ad_{\partial_t}})^i{}_k \wedge \mathrm{d}x^k = \cdots = -f^i{}_{tj} \ \mathrm{d}t \wedge e^j . \tag{4.2.6}$$

Note that taking for instance $\mu(t)$ as in (3.2.14), the corresponding A is not a diffeomorphism and therefore can change topology.

Before giving explicit examples of the construction for the almost abelian case, let us come back to the problem of compactness, which is less straightforward than for nilmanifolds. To this end we need to investigate the monodromy properties of the matrix A_M and the related one-forms under a complete turn around the base circle.

[5] We recall the adjoint action of E is given by $[E, \cdot\ ]$, see appendix B.1.1.

Let us consider the following identification: $t \sim t+t_0$ where t_0 is the periodicity of the base circle. To obtain a consistent construction (having globally defined one-forms) we must preserve the structure of the torus we are fibering over the t direction. This amounts to asking that an arbitrary point of the torus is sent to an equivalent one after we come back to the point t from which we started. The monodromies of the fiber are fixed, thus the only allowed shifts are given by their integer multiples. The way points in the torus are transformed when we go around the base circle is encoded in a matrix $M_{\mathcal{F}}$ which has then to be integer valued. The identification along the t direction is given by

$$T_6 : \begin{cases} t \to t + t_0 \\ x^i \to (M_{\mathcal{F}})^i{}_j x^j \end{cases} \qquad i, j = 1, \ldots, 5\,, \tag{4.2.7}$$

while those along the remaining directions are trivial

$$T_i : \begin{cases} x^i \to x^i + 1 \\ x^j \to x^j \\ t \to t \end{cases} \qquad i, j = 1, \ldots, 5\,;\ i \neq j\,. \tag{4.2.8}$$

Let us now consider the one-forms (4.2.1) we have constructed via the twist A_M. It is straightforward to see that (4.2.1) are invariant under the trivial identifications, while under the non-trivial T_6, we have for $i, j = 1, \ldots, 5$

$$\tilde{e}^i = A_M(t + t_0)^i{}_j \mathrm{d}\tilde{x}^j = [A_M(t) A_M(t_0) M_{\mathcal{F}}]^i{}_j \mathrm{d}x^j\ . \tag{4.2.9}$$

The one-forms are globally defined if they are invariant under this identification:

$$\tilde{e}^i = e^i = A_M(t)^i{}_j \mathrm{d}x^j\,. \tag{4.2.10}$$

Therefore, in the construction, we have to satisfy the following condition:

$$A_M(t_0) M_{\mathcal{F}} = \mathbb{I}_5 \Leftrightarrow M_{\mathcal{F}} = A_M^{-1}(t_0) = A_M(-t_0)\ . \tag{4.2.11}$$

Consistency requires the matrix A_M to be such that $A_M(-t_0)$ is integer valued for at least one $t_0 \neq 0$. This will impose a quantization condition on the period of the base circle, which can take only a discrete set of values (in general it will be a numerable set, as we will see in the examples). Once we fix t_0, the integer entries of $A_M(-t_0)$ will provide the set of identifications.

It is worth stressing that being able to give the correct identifications of the one-forms of the manifold is the same as having a lattice: the identifications (4.2.11) express the lattice action, and give globally defined one-forms only if $A_M(-t_0) = \mu(t_0)$ is integer valued for some t_0. As already discussed in section 3.2, this is the condition to have a lattice (as stated in [37], see also appendix B.1.2). Let us emphasize that the one-forms (4.2.1), constructed via the twist, are globally defined only if we start from a basis of the Lie algebra where $A_M(t)$ is integer valued for some value of t. We give a list of algebras in such a basis in Appendix C.1.2.

As an example, we write the explicit form of the twist matrix for two almost abelian six-dimensional algebras[6] (we already mentioned these two algebras in section 3.3: the corresponding solvmanifolds provide supersymmetric solutions). We also discuss the consistency condition (4.2.11) (globally definedness) for each of these algebras. In the basis where the one-forms are globally defined the two algebras are

$$\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R} \quad : \quad (q_1 25, q_2 15, q_2 45, q_1 35, 0, 0)\,, \tag{4.2.12}$$
$$\mathfrak{g}_{5.17}^{p,-p,\pm 1} \oplus \mathbb{R} \quad : \quad (q_1(p25 + 35), q_2(p15 + 45), q_2(p45 - 15), q_1(p35 - 25), 0, 0)\,. \tag{4.2.13}$$

[6]We use the same notation as in the standard classification of solvable algebras [63, 64, 37]: the number 5 indicates the dimension of the (indecomposable) algebra, while the second simply gives its position in the list of indecomposable algebras of dimension 5.

In both cases the parameters q_1 and q_2 are strictly positive. This is not the most general form of these algebras, which in general contain some free parameters[7] p, q and r. Here we wrote the values of the parameters for which we were able to find a lattice: $p=-q=-r=1$ for the first algebra and $r=\pm 1$ for the second.

In the following, by abuse of notation, we will denote the algebra and the corresponding solvmanifold with the same name.

The algebra (4.2.12) being a direct product of a trivial direction and a five-dimensional indecomposable algebra, the adjoint matrix $ad_{\partial_{x^5}}(\mathfrak{n})$ is block-diagonal, with the non-trivial blocks given by $-ad_{\partial_t}(\mathfrak{n})$ in (3.2.18) and its transpose. Then the twist matrix is

$$A=\begin{pmatrix} A_M & \\ & \mathbb{I}_2 \end{pmatrix} \qquad A_M=\left(\begin{array}{cc|cc} \tilde\alpha & -\tilde\beta & & \\ -\tilde\gamma & \tilde\alpha & & \\ \hline & & \tilde\alpha & -\tilde\gamma \\ & & -\tilde\beta & \tilde\alpha \end{array}\right), \tag{4.2.14}$$

where, not to clutter notation, we defined

$$\tilde\alpha=\cosh(\sqrt{q_1q_2}x^5)\ ,\ \tilde\beta=\sqrt{\frac{q_1}{q_2}}\sinh(\sqrt{q_1q_2}x^5)\ ,\ \tilde\gamma=\sqrt{\frac{q_2}{q_1}}\sinh(\sqrt{q_1q_2}x^5)\ . \tag{4.2.15}$$

The forms obtained by the twist (4.2.14) are globally defined [42]. Indeed they are invariant under constant shifts of each x^i for $i=1,2,3,4$ and 6, with the other variables fixed, and the following non-trivial identification under shifts for x^5

$$(x^1,\dots,x^6)=(\tilde\alpha x^1+\tilde\beta x^2,\tilde\gamma x^1+\tilde\alpha x^2,\tilde\alpha x^3+\tilde\gamma x^4,\tilde\beta x^3+\tilde\alpha x^4,x^5+l,x^6)\,, \tag{4.2.16}$$

where in $\tilde\alpha,\tilde\beta,\tilde\gamma$ we took $x^5=l$. For the above identifications to be discrete [42] $\tilde\alpha$, $\tilde\beta$, and $\tilde\gamma$ must be all integers. This is equivalent to having the matrix $\mu(x^5=l)$ integer and, hence, it is the same as the compactness criterion. The existence of a lattice was also discussed in [29]. There the parameters $\tilde\alpha$, $\tilde\beta$ and $\tilde\gamma$ were set to $\tilde\alpha=2$, $\tilde\beta=3$, $\tilde\gamma=1$.

For the second algebra, $\mathfrak{g}_{5.17}^{p,-p,r}\oplus\mathbb{R}$, we will consider separately the cases $p=0$ and $p\neq 0$. For $p=0$ it reduces to $(q_1 35, q_2 45, -q_2 15, -q_1 25, 0, 0)$ with $r^2=1$. This algebra and the associated manifold have been already mentioned in the previous chapter where it was called s 2.5. For $p\neq 0$ the algebra can be seen as the direct sum

$$\mathfrak{g}_{5.17}^{p,-p,r}\oplus\mathbb{R}\approx s\ 2.5+p(\mathfrak{g}_{5.7}^{1,-1,-1}\oplus\mathbb{R})\,. \tag{4.2.17}$$

The twist matrix is given by

$$A=\begin{pmatrix} A_1A_2 & \\ & \mathbb{I}_2 \end{pmatrix}, \tag{4.2.18}$$

where the two matrices A_1 and A_2 commute and give the two parts of the algebra

$$A_1=\left(\begin{array}{cc|cc} \mathrm{ch} & -\eta_0\,\mathrm{sh} & & \\ -\frac{1}{\eta_0}\mathrm{sh} & \mathrm{ch} & & \\ \hline & & \mathrm{ch} & -\frac{1}{\eta_0}\mathrm{sh} \\ & & -\eta_0\,\mathrm{sh} & \mathrm{ch} \end{array}\right), \qquad A_2=\left(\begin{array}{cc|cc} \mathrm{c} & & -\eta_0\mathrm{s} & \\ & \mathrm{c} & & -\frac{1}{\eta_0}\mathrm{s} \\ \hline \frac{1}{\eta_0}\mathrm{s} & & \mathrm{c} & \\ & \eta_0\mathrm{s} & & \mathrm{c} \end{array}\right), \tag{4.2.19}$$

[7] The general form for $\mathfrak{g}_{5.7}^{p,q,r}$ is

$$\frac{1}{2}\Big(-\beta_0(1+r)15+q_1(1-r)25,-\beta_0(1+r)25+q_2(1-r)15,-\beta_0(q+p)35+q_2(p-q)45,-\beta_0(q+p)45+q_1(p-q)35,0\Big)\,,$$

where we set $\beta_0=\sqrt{q_1q_2}$. Similarly, for $\mathfrak{g}_{5.17}^{p,-p,r}$ we have

$$\Big(q_1p25+\frac{1}{2}[q_1(r^2+1)35+\beta_0(r^2-1)45],q_2p15+\frac{1}{2}[q_2(r^2+1)45+\beta_0(r^2-1)35],q_2(-15+p45),q_1(-25+p35),0\Big)\,.$$

where now we define $\eta_0 = \sqrt{\frac{q_1}{q_2}}$ and

$$\begin{aligned} \mathrm{ch} &= \cosh(p\sqrt{q_1 q_2} x^5) \qquad & \mathrm{c} &= \cos(\sqrt{q_1 q_2} x^5) \\ \mathrm{sh} &= \sinh(p\sqrt{q_1 q_2} x^5) \qquad & \mathrm{s} &= \sin(\sqrt{q_1 q_2} x^5) \, . \end{aligned}$$

In this case, imposing that the forms given by the twist (4.2.18) are globally defined under discrete identifications fixes the parameters in the twist to (with $x^5 = l$)

$$\begin{aligned} &\mathrm{ch\ c} = n_1 \ , \ \eta_0 \ \mathrm{sh\ c} = n_2 \ , \ \frac{1}{\eta_0}\mathrm{sh\ c} = n_3 \\ &\mathrm{sh\ s} = n_4 \ , \ \eta_0 \ \mathrm{ch\ s} = n_5 \ , \ \frac{1}{\eta_0}\mathrm{ch\ s} = n_6 \ , \quad n_i \in \mathbb{Z} \ . \end{aligned} \tag{4.2.20}$$

The equations above have no solutions if the integers n_i are all non-zero. The only possibilities are either $n_1 = n_2 = n_3 = 0$ or $n_4 = n_5 = n_6 = 0$ (plus the case where all are zero, which is of no interest here). If one also imposes that the constraints must be solved both for $p = 0$ and $p \neq 0$, the first option, $n_1 = n_2 = n_3 = 0$, has to be discarded and the only solution is

$$\begin{aligned} &n_4 = n_5 = n_6 = 0 \ , \ s = 0 \ , \ l = \frac{k\ \pi}{\sqrt{q_1 q_2}} \ , \ c = (-1)^k \ , \ \tilde{n}_1 = (-1)^k n_1 > 0 \ , \ k \in \mathbb{Z} \\ &\mathrm{ch} = \tilde{n}_1 \ , \ \mathrm{sh}^2 = n_2 n_3 \ , \ n_3 \eta_0^2 = n_2 \ , \ n_2 n_3 = \tilde{n}_1^2 - 1 \ , \ p = \frac{\cosh^{-1}(\tilde{n}_1)}{k\ \pi} \ . \end{aligned} \tag{4.2.21}$$

p is quantized by two integers, but one can show that it can be as close as we want to any real value (the ensemble is dense in $\mathbb{R}$). It can be restricted to be positive for the $\cosh^{-1}$ to be defined.

4.3 Embedding the twist in GCG

In section 2.3, we introduced some notions of GCG and discussed $O(d,d)$ transformations in this context. We showed that, given a manifold M with a metric g and a B-field, one can introduce the generalized metric $\mathcal{H}$, or equivalently the generalized vielbein $\mathcal{E}$, on the generalized tangent bundle E. A length element on E is given by $\mathcal{E}\mathrm{d}X$, where

$$\mathrm{d}X^M = \begin{pmatrix} \mathrm{d}x^m \\ \partial_m \end{pmatrix} \ , \ m = 1 \dots d \ . \tag{4.3.1}$$

So, in this formalism, it is natural to extend a transformation acting on the forms $\mathrm{d}x^m$ to a transformation acting on the full $\mathrm{d}X$. Therefore, we naturally embed the twist transformation given in (4.2.1) into an $O(d,d)$ transformation given by

$$O = \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} \ . \tag{4.3.2}$$

We will now rather act on the generalized vielbein

$$\mathcal{E} \mapsto \mathcal{E}' = \mathcal{E} O \ . \tag{4.3.3}$$

As discussed in the introduction, the twist transformation can be further extended. The transformation (4.3.2) being local, let us consider a more general local $O(d,d)$ transformation. We should restrict ourselves to the full subgroup of $O(d,d)$ transformations that preserve the lower triangular form (2.3.23) of the generalized vielbein, as discussed in section 2.3. Therefore, the transformation O also has to be lower triangular:

$$O = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \ , \tag{4.3.4}$$

where A, C, and D are $d \times d$ blocks. The $O(d,d)$ constraints reduce in this case to

$$A^T C + C^T A = 0 \qquad\qquad A^T D = \mathbb{I}\ . \tag{4.3.5}$$

This fixes the matrix D to be the inverse of A^T: $D = A^{-T}$. The resulting transformation can be written as

$$O = \begin{pmatrix} A & 0 \\ C & A^{-T} \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ X & \mathbb{I} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -Y & \mathbb{I} \end{pmatrix}\ , \tag{4.3.6}$$

with $Y = A^T X A - A^T C$ for any X. So the general twist transformation is a product of B-transforms (given by X and Y) and a $GL(d)$ transformation given by A. The A matrix can correspond to the topological transformation discussed previously as the action on one-forms[8], or could contain some additional ingredients like diagonal blocks which will rescale the metric. The purely topological twist transformations previously discussed all had $\det(A) = 1$. Therefore, according to (2.3.34), only a scaling of the metric would transform the dilaton.

Let us now be more specific, and consider the case where M is a torus fibration (with connection), $T^n \hookrightarrow M \xrightarrow{\pi} \mathcal{B}$, and a twist transformation of the type

$$O = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} A_{\mathcal{B}} & 0 & 0 & 0 \\ A_{\mathcal{C}} & A_{\mathcal{F}} & 0 & 0 \\ C_{\mathcal{B}} & C_{\mathcal{C}} & D_{\mathcal{B}} & D_{\mathcal{C}} \\ C_{\mathcal{C}'} & C_{\mathcal{F}} & 0 & D_{\mathcal{F}} \end{pmatrix}\ . \tag{4.3.7}$$

In the second matrix, we split the base ($\mathcal{B}$), fiber ($\mathcal{F}$) and mixed elements. As discussed previously, the $O(d,d)$ constraints fix D as

$$D = A^{-T} = \begin{pmatrix} A_{\mathcal{B}}^{-T} & -A_{\mathcal{B}}^{-T} A_{\mathcal{C}}^T A_{\mathcal{F}}^{-T} \\ 0 & A_{\mathcal{F}}^{-T} \end{pmatrix}\ , \tag{4.3.8}$$

and in addition allow to parametrize C in terms of three unconstrained matrices $\tilde{C}_{\mathcal{B}}$, $\tilde{C}_{\mathcal{F}}$, $\tilde{C}_{\mathcal{C}}$,

$$C = \begin{pmatrix} A_{\mathcal{B}}^{-T}(\tilde{C}_{\mathcal{B}} - A_{\mathcal{C}}^T A_{\mathcal{F}}^{-T} \tilde{C}_{\mathcal{C}}) & -A_{\mathcal{B}}^{-T}(\tilde{C}_{\mathcal{C}}^T + A_{\mathcal{C}}^T A_{\mathcal{F}}^{-T} \tilde{C}_{\mathcal{F}}) \\ A_{\mathcal{F}}^{-T} \tilde{C}_{\mathcal{C}} & A_{\mathcal{F}}^{-T} \tilde{C}_{\mathcal{F}} \end{pmatrix}\ , \tag{4.3.9}$$

with $\tilde{C}_{\mathcal{B}}$ and $\tilde{C}_{\mathcal{F}}$ anti-symmetric.

Later on, we shall study when and how the transformation (4.3.7) maps one string background to another. In general, two internal manifolds connected in this way will have different topologies. Typically such topology changes are associated with large transformations, while (4.3.7) is connected to the identity. The topological properties of related backgrounds are determined by the global properties of the matrices C and $A_{\mathcal{C}}$.

Actually, the off-diagonal piece $A_{\mathcal{C}}$ of the $GL(d)$ part corresponds to the topological transformation discussed for nilmanifolds with only one fibration, as one can see in formula (4.2.3). The diagonal pieces $A_{\mathcal{B}}$ and $A_{\mathcal{F}}$ will then give additional scalings of the metric as discussed.

4.3.1 Action on the generalized vielbein

One reason to introduce the transformation (4.3.7) is to relate spaces that are direct products of two manifolds, for instance $T^4 \times T^2$, into spaces that are non-trivial fibrations with connection. To see how this is achieved we can look at the $O(d,d)$ transformations of the generalized vielbein.

[8]Note that the transformations considered as examples of (4.2.1) were all connected to the identity, so are indeed elements of $GL(d)$.

We will be interested in solutions where the manifold M is a n-dimensional torus fibration (with coordinates y^m) over a base $\mathcal{B}$ (with coordinates $x^{\tilde{m}}$)

$$\mathrm{d}s^2 = g_{\tilde{m}\tilde{n}}\mathrm{d}x^{\tilde{m}}\mathrm{d}x^{\tilde{n}} + g_{mn}(\mathrm{d}y^m + A^m{}_{\tilde{r}}\mathrm{d}x^{\tilde{r}})(\mathrm{d}y^n + A^n{}_{\tilde{s}}\mathrm{d}x^{\tilde{s}}) \ . \tag{4.3.10}$$

where A is here the connection of the fibration. The corresponding vielbein and one-forms are

$$\begin{aligned} e^{\tilde{a}} &= e^{\tilde{a}}{}_{\tilde{m}}\mathrm{d}x^{\tilde{m}} && (4.3.11)\\ e^{a} &= e^{a}{}_{m}(\mathrm{d}y^m + A^m{}_{\tilde{n}}\mathrm{d}x^{\tilde{n}}) = e^{a}{}_{m}\Theta^m \ , && (4.3.12) \end{aligned}$$

where $\tilde{a}$ and a are the local Lorentz indices on the base and the fiber respectively, while $\tilde{m}$ and m are the corresponding target-space indices. We take also a non-trivial B-field of the form

$$\begin{aligned} B &= B^{(2)} + B^{(1)} + B^{(0)} \\ &= \frac{1}{2}B_{\tilde{m}\tilde{n}}\ \mathrm{d}x^{\tilde{m}} \wedge \mathrm{d}x^{\tilde{n}} + B_{\tilde{m}m}\ \mathrm{d}x^{\tilde{m}} \wedge \Theta^m + \frac{1}{2}B_{mn}\ \Theta^m \wedge \Theta^n \ , \end{aligned} \tag{4.3.13}$$

where $B^{(2)}$ is the component entirely on the base, $B^{(1)}$ has one component on the base and one on the fiber, and $B^{(0)}$ is on the fiber

$$\begin{aligned} B^{(2)} &= \frac{1}{2}(B_{\tilde{m}\tilde{n}} - 2B_{m[\tilde{m}}A^m{}_{\tilde{n}]} + B_{mn}A^m{}_{\tilde{m}}A^n{}_{\tilde{n}})\ \mathrm{d}x^{\tilde{m}} \wedge \mathrm{d}x^{\tilde{n}} \ , && (4.3.14)\\ B^{(1)} &= (B_{\tilde{m}m} - B_{mn}A^n{}_{\tilde{m}})\ \mathrm{d}x^{\tilde{m}} \wedge \mathrm{d}y^m \ , && (4.3.15)\\ B^{(0)} &= \frac{1}{2}B_{mn}\ \mathrm{d}y^m \wedge \mathrm{d}y^n \ . && (4.3.16) \end{aligned}$$

The generalized vielbein in (2.3.24) then take the form

$$\mathcal{E}^A{}_M\mathrm{d}X^M = \begin{pmatrix} e & 0 \\ -\hat{e}B & \hat{e} \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \partial \end{pmatrix} = \begin{pmatrix} e^{\tilde{a}}{}_{\tilde{m}} & 0 & 0 & 0 \\ A^a{}_{\tilde{m}} & e^a{}_m & 0 & 0 \\ -B_{\tilde{a}\tilde{m}} & -B_{\tilde{a}m} & \hat{e}_{\tilde{a}}{}^{\tilde{m}} & \hat{A}_{\tilde{a}}{}^m \\ -B_{a\tilde{m}} & -B_{am} & 0 & \hat{e}_a{}^m \end{pmatrix} \begin{pmatrix} \mathrm{d}x^{\tilde{m}} \\ \mathrm{d}y^m \\ \partial_{\tilde{m}} \\ \partial_m \end{pmatrix} \ , \tag{4.3.17}$$

where $\hat{e} = (e^{-1})^T$. To simplify the notation we defined the connections $A^a{}_{\tilde{n}} = e^a{}_m A^m{}_{\tilde{n}}$ and $\hat{A}_{\tilde{a}}{}^m = -\hat{e}_{\tilde{a}}{}^{\tilde{m}}A_{\tilde{m}}{}^m$. Similarly the components of the B-field are

$$\begin{aligned} B_{\tilde{a}n} &= \hat{e}_{\tilde{a}}{}^{\tilde{m}}B_{\tilde{m}n} \qquad & B_{\tilde{a}\tilde{n}} &= \hat{e}_{\tilde{a}}{}^{\tilde{m}}(B_{\tilde{m}\tilde{n}} + B_{\tilde{m}m}A^m{}_{\tilde{n}} - A_{\tilde{m}}{}^m B_{m\tilde{n}}) \ , && (4.3.18)\\ B_{an} &= \hat{e}_{\tilde{a}}{}^m B_{mn} \qquad & B_{a\tilde{n}} &= \hat{e}_a{}^m(B_{mn}A^n{}_{\tilde{n}} + B_{n\tilde{n}}) \ . && (4.3.19) \end{aligned}$$

As an example of the transformation (4.3.7), consider now a manifold which is a direct product of a base and a “fiber” and with no B-field. The generalized vielbein take the simple form

$$\mathcal{E} = \begin{pmatrix} e_{\mathcal{B}} & 0 & 0 & 0 \\ 0 & e_{\mathcal{F}} & 0 & 0 \\ 0 & 0 & \hat{e}_{\mathcal{B}} & 0 \\ 0 & 0 & 0 & \hat{e}_{\mathcal{F}} \end{pmatrix} \ , \tag{4.3.20}$$

where with obvious notation $e_{\mathcal{B}}$ and $e_{\mathcal{F}}$ denote the vielbein on the base and the fiber. After the transformation (4.3.7), it becomes

$$\mathcal{E}' = \begin{pmatrix} e_{\mathcal{B}}A_{\mathcal{B}} & 0 & 0 & 0 \\ e_{\mathcal{F}}A_{\mathcal{C}} & e_{\mathcal{F}}A_{\mathcal{F}} & 0 & 0 \\ \hat{e}_{\mathcal{B}}C_{\mathcal{B}} & \hat{e}_{\mathcal{B}}C_{\mathcal{C}} & \hat{e}_{\mathcal{B}}D_{\mathcal{B}} & \hat{e}_{\mathcal{B}}D_{\mathcal{C}} \\ \hat{e}_{\mathcal{F}}C_{\mathcal{C}'} & \hat{e}_{\mathcal{F}}C_{\mathcal{F}} & 0 & \hat{e}_{\mathcal{F}}D_{\mathcal{F}} \end{pmatrix} \ . \tag{4.3.21}$$

Comparing the previous expression with (4.3.17), it is easy to see that the new background has a non-trivial B-field

$$B' = -A^T C = -\begin{pmatrix} \tilde{C}_{\mathcal{B}} & -\tilde{C}_{\mathcal{C}}^T \\ \tilde{C}_{\mathcal{C}} & \tilde{C}_{\mathcal{F}} \end{pmatrix} , \tag{4.3.22}$$

and a non-trivial fibration structure with connection[9] $A' = A_{\mathcal{F}}^{-1} A_{\mathcal{C}}$. The transformed metric is then

$$\mathrm{d}s^2 = g'_{\tilde{m}\tilde{n}} \mathrm{d}x^{\tilde{m}} \mathrm{d}x^{\tilde{n}} + g'_{mn} (\mathrm{d}y^m + A'^m{}_{\tilde{r}} \mathrm{d}x^{\tilde{r}})(\mathrm{d}y^n + A'^n{}_{\tilde{s}} \mathrm{d}x^{\tilde{s}}) , \tag{4.3.23}$$

where $g'_{\tilde{m}\tilde{n}} = (A_{\mathcal{B}}^T \, g_{\mathcal{B}} \, A_{\mathcal{B}})_{\tilde{m}\tilde{n}}$ and $g'_{mn} = (A_{\mathcal{F}}^T \, g_{\mathcal{F}} \, A_{\mathcal{F}})_{mn}$. From the explicit form of the metrics g and g', (4.3.23), the transformed dilaton (2.3.34) becomes

$$e^{\phi'} = e^{\phi} |\det(A_{\mathcal{B}}) \ \det(A_{\mathcal{F}})|^{\frac{1}{2}} . \tag{4.3.24}$$

The matrices $A_{\mathcal{B}}$, $A_{\mathcal{F}}$, $A_{\mathcal{C}}$, $\tilde{C}_{\mathcal{B}}$, $\tilde{C}_{\mathcal{F}}$, and $\tilde{C}_{\mathcal{C}}$ are completely arbitrary, and hence the transformation (4.3.7) allows to go from whatever metric, dilaton and B-field, to any other metric, dilaton, connection, and B-field.

4.3.2 Action on pure spinors

In section 2.3.3 we discussed the action of an $O(d,d)$ in the spinorial representation on the pure spinors $\Psi_\pm$ (2.4.13) of E. The twist transformation discussed previously can be written in this way. If we consider only a $GL(d)$ transformation as in (4.3.2), and furthermore consider for A only the action on the forms discussed in the almost abelian case (4.2.5), then the twist action on the spinor reads

$$O \cdot \Psi = \frac{1}{\sqrt{\det A}} \, e^{-t \ [ad_{\partial_t}(\mathfrak{n})]^m{}_n e^n \wedge \iota_m} \cdot \Psi , \tag{4.3.25}$$

where $e^{m=1\dots6}$ is a given basis of one-forms on M, and ι_m the associated contraction.

Let us consider the more general action given by the triple product (4.3.6). It can be seen as a succession of a B-transform, a $GL(d)$ action and another B-transform. This leads to the following expression for the $O(d,d)$ action on the spinors

$$O = \frac{1}{\sqrt{\det A}} e^{-\frac{1}{2} y_{mn} \mathrm{d}x^m \wedge \mathrm{d}x^n} \ e^{a^m{}_n \mathrm{d}x^n \wedge \ \iota_{\partial_m}} \ e^{\frac{1}{2} x_{mn} \mathrm{d}x^m \wedge \mathrm{d}x^n} . \tag{4.3.26}$$

Since $O(d,d)$ acts on the generalized vielbein from the right and on pure spinors from the left, we have exchanged the order of the transformations with respect to (4.3.6). From this expression, we can easily read the transformation of the dilaton which is present in the pure spinors $\Psi_\pm$ (2.4.13): it is transformed by the factor $\sqrt{\det A}$ as required in an $O(d,d)$ transformation.

We recall that $Y = A^T X A - A^T C$ for any given X. In the transformation of the generalized vielbein, we showed that the B-field of the transformed background is $B' = A^T B A - A^T C$. Therefore we can interpret X as the B-field of the original solution and Y as the new one. Then we can read from the definition of pure spinors $\Psi_\pm$ (2.4.13) that the action (4.3.26) will simply erase the old B-field B and replace it by the new one B' after the $GL(d)$ transformation.

Let us now apply this twist transformation to six-dimensional manifolds M which are T^2 fibration (with connection) over a four-dimensional base $\mathcal{B}$

$$\mathcal{B} \times T^2 \Rightarrow T^2 \hookrightarrow M \xrightarrow{\pi} \mathcal{B} . \tag{4.3.27}$$

The $GL(d)$ part of the transformation is that of (4.3.7). Depending on the particular example, we take $\mathcal{B}$ to be T^4, or $K3$. We will denote the holomorphic coordinate on the fiber by $z = \theta^1 + i\theta^2$.

[9] We recover the fact that the off-diagonal piece $A_{\mathcal{C}}$ provides a connection when starting from none, as in (4.2.3).

Then the torus generators are defined as ∂_z and $\partial_{\bar{z}}$, and the connection one-forms by $\Theta^I = \mathrm{d}\theta^I + A^I$, with $\Theta = \Theta^1 + i\Theta^2$ and $\alpha = A^1 + iA^2$. The fibration will be in general non-trivial, and the curvature two-forms $F^I \in \Omega^2_{\mathbb{Z}}(\mathcal{B})$ are given by $\mathrm{d}\Theta^I = \pi^* F^I$.

Our starting point is a trivial T^2 fibration. For simplicity, we set the B-field to zero and the dilaton to a constant. The pure spinors are as in (2.4.15) and (2.4.16), with the $SU(3)$ structure defined by

$$J = J_{\mathcal{B}} + \frac{i}{2} g_{z\bar{z}} \mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{4.3.28}$$

$$\Omega = \sqrt{|g|}\ \omega_{\mathcal{B}} \wedge \mathrm{d}z\ , \tag{4.3.29}$$

where $|g|$ is the determinant of the metric on the torus fiber, $J_{\mathcal{B}}$ and $\omega_{\mathcal{B}}$ the Kähler and holomorphic two-forms on the base.

In the transformation (4.3.26) we set $x_{mn} = 0$ since there is no initial B-field, and take y_{mn} an arbitrary antisymmetric matrix. This will act as a standard B-transform giving the new B-field. Here we will concentrate on the $GL(6)$ part. For simplicity, we take the action on the base to be trivial

$$A = \begin{pmatrix} 1_4 & 0 \\ A_{\mathcal{C}} & A_{\mathcal{F}} \end{pmatrix} = \begin{pmatrix} 1_4 & 0 \\ 0 & A_{\mathcal{F}} \end{pmatrix} \begin{pmatrix} 1_4 & 0 \\ A' & 1_2 \end{pmatrix} \tag{4.3.30}$$

and

$$A_{\mathcal{F}} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}\ , \qquad A' = A_{\mathcal{F}}^{-1} A_{\mathcal{C}} = \begin{pmatrix} A^1{}_{\tilde{m}} \\ A^2{}_{\tilde{n}} \end{pmatrix}\ . \tag{4.3.31}$$

Note that we take an action as those constructing the nilmanifolds (4.2.3), and additionally a scaling of the metric. With this choice, the $GL(6)$ factor in (4.3.26) becomes

$$O_a = \frac{1}{\sqrt{\det A_{\mathcal{F}}}}\ e^{A^1\ \iota_{\partial_1} + A^2\ \iota_{\partial_2}}\ e^{\lambda_1\ \mathrm{d}x^1 \wedge\ \iota_{\partial_1} + \lambda_2 \mathrm{d}x^2 \wedge\ \iota_{\partial_2}}\ , \tag{4.3.32}$$

with $A^I = A^I{}_{\tilde{m}} \mathrm{d}x^{\tilde{m}}$ for $I = 1, 2$. In terms of the complex connection α, the off-diagonal block becomes

$$\begin{aligned} A^I\ \iota_{\partial_I} &= \alpha \wedge i_{\partial z} + \overline{\alpha} \wedge i_{\partial \overline{z}} \\ e^{A^I\ \iota_{\partial_I}} &= 1 + (\alpha \wedge i_{\partial z} + \overline{\alpha} \wedge i_{\partial \overline{z}}) + \alpha \wedge \overline{\alpha} \wedge i_{\partial \overline{z}}\ i_{\partial z} = 1 + o\cdot\ , \end{aligned} \tag{4.3.33}$$

where $o\cdot$ sends a form to another form with same degree. The diagonal blocks give

$$e^{\lambda_I \mathrm{d}x^I \wedge\ \iota_{\partial_I}} = \prod_{I=1,2} \left[1 + (e^{\lambda_I} - 1) \mathrm{d}x^I \wedge\ \iota_{\partial_I} \right]\ . \tag{4.3.34}$$

To derive (4.3.34) we used the fact the operators $\mathrm{d}x^I \wedge\ \iota_{\partial_I}$ commute for different values of I, and $(\mathrm{d}x^I \wedge\ \iota_{\partial_I})^k = \mathrm{d}x^I \wedge\ \iota_{\partial_I}$.

The effect of (4.3.34) on Ω and J is to rescale the fiber components, while (4.3.33) introduces the shift of the fiber direction by the connections α and $\overline{\alpha}$

$$J' = J_{\mathcal{B}} + \frac{i}{2} g'_{z\overline{z}}\ \Theta \wedge \overline{\Theta} \tag{4.3.35}$$

$$\Omega' = \sqrt{|g'|}\ \omega_{\mathcal{B}} \wedge \Theta\ , \tag{4.3.36}$$

where $g'_{z\overline{z}} = e^{2\lambda} g_{z\overline{z}}$ and $\Theta = \mathrm{d}z + \alpha$. In order not to change the complex structure, we have to set $\lambda_1 = \lambda_2 = \lambda$. Finally, from (4.3.33) and (4.3.34), it is straightforward to compute the new pure spinors

$$\begin{aligned} \Psi_+ &= e^{i\theta_+} e^{-\phi}\ e^{-iJ} &\longrightarrow\quad \Psi'_+ &= e^{i\theta_+} e^{-\phi'} e^{-B'} e^{-iJ'}\ , \\ \Psi_- &= -i e^{i\theta_-} e^{-\phi}\ \Omega &\longrightarrow\quad \Psi'_- &= -i e^{i\theta_-} e^{-\phi'} e^{-B'} \Omega'\ . \end{aligned} \tag{4.3.37}$$

Here we took the normalized pure spinors and we did not transform the phases (this possibility was mentioned in section 2.3.3 and will be used later). The new B-field is clearly $B' = \frac{1}{2} y_{mn} \mathrm{d}x^m \wedge \mathrm{d}x^n$ and the dilaton is transformed by the trace part of the $GL(6)$ transformation

$$e^{-\phi'} = (\det A_{\mathcal{F}})^{-1/2} e^{-\phi} = e^{-\lambda} e^{-\phi} \,. \tag{4.3.38}$$

4.4 Type II transformations

We would like to use the $O(6,6)$ transformation discussed in the previous section as a solution generating technique in type II SUGRA: starting from a known solution, we would like to get a new one. Instead of studying when the transformation preserves the equations of motion, we will focus on the supersymmetry conditions and derive the constraints imposed by SUSY on the $O(6,6)$ transformation in order for the latter to map solutions to new ones. We will not solve these constraints in full generality. Here we concentrate on an explicit application of our transformation to the context of $SU(3)$ structure compactifications on T^6 or six-dimensional twisted tori.

4.4.1 Generating solutions: constraints in type II and RR fields transformations

We consider type II backgrounds corresponding to warp products of four-dimensional Minkowski times a six-dimensional compact manifold, as described in section 2.2.1 and we use the pure spinors $\Psi_\pm$ on E defined in (2.4.13). The supersymmetry conditions (2.4.18), (2.4.19), and (2.4.20) can be rewritten in terms these pure spinors as

$$\begin{aligned}
&\mathrm{d}(e^{3A}\Psi_1) = 0\,,\\
&\mathrm{d}(e^{2A}\,\mathrm{Re}\,\Psi_2) = 0\,,\\
&\mathrm{d}(e^{4A}\,\mathrm{Im}\,\Psi_2) = e^{4A} e^{-B} * \lambda(F) = R\,,
\end{aligned} \tag{4.4.1}$$

where we introduce the notation R for later use, and we took $|a|^2 = e^A$. Obviously, $\Psi_1 = \Psi_{+/-}$ and $\Psi_2 = \Psi_{-/+}$ in IIA/B.

Consider now a solution of the supersymmetry equations and Bianchi identities, and apply to the associated pure spinors the complex transformation (2.3.44)

$$O_c^{\pm} = e^{i\theta_c^{\pm}} O \qquad \Rightarrow \qquad \Psi'_\pm = O_c^\pm \, \Psi_\pm \,, \tag{4.4.2}$$

where O is a real $O(d,d)$ transformation in the spinorial representation. We want to determine what are the conditions on O_c in order to get a new solution (at least to the SUSY equations). The preservation of the closure equations will provide at least $\mathcal{N}=1$ supersymmetry, while the action of the transformation on the rest of the fields is then used to define the transformed RR fields. Indeed, we do not know how to transform directly the RR fields under $O(d,d)$, so we will read the new ones out of the new SUSY equations.

The conditions to get new solutions are easily determined by imposing that the transformed pure spinors are again solutions of the SUSY equations

$$\begin{aligned}
\mathrm{d}(e^{3A}\Psi'_1) &= 0\\
\mathrm{d}(e^{2A}\mathrm{Re}\Psi'_2) &= 0\\
\mathrm{d}(e^{4A}\mathrm{Im}\Psi'_2) &= R'\,,
\end{aligned} \tag{4.4.3}$$

where R' gives the new RR fields. Then expanding into real and imaginary parts, we obtain

$$\begin{aligned}
&\mathrm{d}(O)\,\Psi_1 = 0\\
&\cos\theta_c^+\,\mathrm{d}(O)\,e^{2A}\,\mathrm{Re}\,\Psi_2 - \sin\theta_c^+\,\mathrm{d}(e^{-2A}O)\,e^{4A}\,\mathrm{Im}\,\Psi_2 = e^{-2A}\sin\theta_c^+\,O\,R\\
&\sin\theta_c^+\,\mathrm{d}(e^{2A}O)\,e^{2A}\,\mathrm{Re}\,\Psi_2 + \cos\theta_c^+\,\mathrm{d}(O)\,e^{4A}\,\mathrm{Im}\,\Psi_2 = R' - \cos\theta_c^+\,O\,R\,.
\end{aligned} \tag{4.4.4}$$

The last equation defines the transformed RR field

$$R' = \cos(\theta_c^+) O\, R + \sin(\theta_c^+) \mathrm{d}(e^{2A} O)\, e^{2A}\, \mathrm{Re}\, \Psi_2 + \cos(\theta_c^+) \mathrm{d}(O)\, e^{4A}\, \mathrm{Im}\, \Psi_2 \ . \tag{4.4.5}$$

Because of the phase, the last equations are actually correct in IIB. In IIA, one has to replace θ_c^+ by θ_c^-. The first two equations of (4.4.4) are the constraints the O_c transformation has to fulfill in order to map solutions to new solutions of type II supergravity. Out of the third equation, one still has to check the BI. As already mentioned at the beginning of this section, we do not analyse in general the system of constraints above, but we will do it for particular kinds of operator O.

An interesting feature of this transformation is the possible mixing between the NSNS and RR sectors, as we can see in (4.4.5). This is due to the locality of the operator O, and to the complexification of the $O(6,6)$ transformation by the $U(1)$ action on the line of pure spinors. Note also that such a complexification is necessary to relate different types of sources.

4.4.2 Relating solutions on nilmanifolds

In this section, we will use our twist transformation (4.3.7) to relate compactifications on T^6 to nilmanifolds that are fibrations of T^2 over T^4

$$T^4 \times T^2 \Rightarrow T^2 \hookrightarrow M \stackrel{\pi}{\longrightarrow} T^4 \, . \tag{4.4.6}$$

At the end of the section, we will consider a further twist to construct a second fibration.

The construction of a nilmanifold from a torus (4.2.3) could be extended in GCG into (4.3.7), which transforms, in addition to the topology, the metric, the B-field and the dilaton. As shown in section 4.3.2, the same transformation can be performed on the pure spinors and then the transformation of the RR sector is read indirectly from the SUSY equation (4.4.5). The main result of our analysis is to show that the nilmanifold solutions listed in section 3.3 correspond to the only possible twists from a T^6.

Nilmanifolds with only one fibration

We will first consider the nilmanifolds wich consist of a single fibration. We will denote as previously the torus generators by ∂_z and $\partial_{\bar{z}}$, the connection one-forms by $\Theta^I = \mathrm{d}\theta^I + A^I$ and the curvature two-forms by F^I with $I = 1, 2$.

The twist transformations necessarily relate manifolds with different topological properties. This can be seen by computing the Betti numbers of the different manifolds. For the direct product of T^2 with a generic base $\mathcal{B}$, the Betti numbers are

$$\begin{aligned} b_1 &= b_1(\mathcal{B}) + 2 \ , \\ b_2 &= b_2(\mathcal{B}) + 2b_1(\mathcal{B}) + 1 \, , \\ b_3 &= b_3(\mathcal{B}) + 2b_2(\mathcal{B}) + b_1(\mathcal{B}) \, . \end{aligned} \tag{4.4.7}$$

Clearly the Betti numbers for generic M are smaller than for $\mathcal{B}\times T^2$ and will depend on the topological properties of the curvature F^I. Indeed as $\mathrm{d}\theta^I$ is mapped to $\Theta^I = \mathrm{d}\theta^I + A^I$ and $\mathrm{d}\Theta^I = \pi^* F^I$, the two one-forms $\mathrm{d}\theta^I$, which were non-trivial in cohomology, are replaced by forms that are not closed. At the same time the two closed two-forms F^I, while being non-trivial in cohomology on $\mathcal{B}$, are trivial in cohomology on M. When the base is T^4, we find $b_1(M) = b_1(T^4) = 4$. There are only seven classes of nilmanifolds with $b_1 = 4$. It is not hard to check that three of them are actually affine T^2 fibrations over T^4 (circle fibrations over five-manifolds which are in turn circle fibrations over T^4)

n 4.1	(0, 0, 0, 0, 12, 15+ 34)	$M = I_6$
n 4.2	(0, 0, 0, 0, 12, 15)	$M = T^2 \times I_4$
n 4.3	(0, 0, 0, 0, 12, 14+ 25)	$M = S^1 \times I_5$

where I_{n+2} is a sequence of two circle fibrations over T^n. This leaves us with four topologically distinct cases of two commuting $U(1)$ fibrations[10] over T^4

n 4.5	$(0,0,0,0,12,34)$	$M = N_3 \times N_3$	$b_2(M) = 8$	$b_3(M) = 10$
n 4.6	$(0,0,0,0,13,14)$	$M = S^1 \times N_5$	$b_2(M) = 9$	$b_3(M) = 12$
n 4.4	$(0,0,0,0,2\times 13,14+23)$	$M = N_6^{(1)}$	$b_2(M) = 8$	$b_3(M) = 10$
n 4.7	$(0,0,0,0,13+42,14+23)$	$M = N_6^{(2)}$	$b_2(M) = 8$	$b_3(M) = 10$

where N_3 is a circle fibration over T^2, N_5 is a T^2 fibration over T^3 and $N_6^{(1)}$ and $N_6^{(2)}$ are two distinct T^2 fibrations over T^4.

Type C solutions, i.e. solutions with a non-trivial RR F_3 with O5/D5 sources, can be obtained on some of these manifolds by two T-dualities along the fiber from a type B solution on T^6. The latter has a non-trivial five-form which is related to the warp factor and an imaginary anti-self dual complex three-form flux $g_s F_3 = -*H$. According to standard Buscher rules, the components of the B-field with one leg along the fiber give, after T-duality, the non-trivial connections. Under T-duality the O3 planes are mapped to O5 planes.

Here we shall show that solutions on such manifolds can also be related via our twist transformation (4.3.7) to T^6 with O3 planes, a non-trivial five-form flux F_5 and a trivial NSNS flux (and therefore not T-dual). In this background the five-form flux is related to the warp factor

$$g_s F_5 = e^{4A} * \mathrm{d}(e^{-4A}) \,, \tag{4.4.8}$$

while the dilaton is constant $e^{\phi} = g_s$. All other fluxes are zero. The complex structure is chosen as

$$\begin{aligned} \chi^1 &= e^1 + ie^2 \,, \\ \chi^2 &= e^3 + ie^4 \,, \\ \chi^3 &= e^5 + ie^6 \,, \end{aligned} \tag{4.4.9}$$

where χ^i are one-forms and the one-forms on the torus are $e^m = e^{-A}\mathrm{d}x^m$ with $m = 1, \ldots, 6$. Then the $SU(3)$ structure and the corresponding pure spinors are

$$\Omega = \chi^1 \wedge \chi^2 \wedge \chi^3 \qquad\qquad \Psi_- = -\frac{i}{g_s}\Omega \tag{4.4.10}$$

$$J = \frac{i}{2}\chi^i \wedge \overline{\chi}^i \qquad\qquad \Psi_+ = \frac{i}{g_s}\, e^{-iJ} \,. \tag{4.4.11}$$

The O3 projection fixes one phase $\theta_+ = \frac{\pi}{2}$, while we choose for the other $\theta_- = 0$.

The idea is now to apply the transformations (2.3.44) and (4.3.32) to the previous solution and see under which conditions we can reproduce the nilmanifolds n 4.4, n 4.5, n 4.6, n 4.7. We choose the T^2 torus fiber in the directions x^5 and x^6. Since we are connecting solutions with zero NSNS flux, we do not bother to consider the contribution of the B-transform. The new pure spinors are given by (4.3.37)

$$\begin{aligned} \Psi'_- &= -ie^{i\theta_c^-}\, e^{-\phi'}\,\Omega' \\ \Psi'_+ &= ie^{i\theta_c^+}\, e^{-\phi'} e^{-iJ'} \,, \end{aligned} \tag{4.4.12}$$

where the $SU(3)$ structure takes the form (4.3.35)

$$J' = J_{\mathcal{B}} + \frac{i}{2} g'_{z\overline{z}}\Theta \wedge \overline{\Theta} \tag{4.4.13}$$

$$\Omega' = \sqrt{|g'|}\, \omega_{\mathcal{B}} \wedge \Theta \,, \tag{4.4.14}$$

[10]Note that here we label the nilmanifolds as in section 3.3, but for n 4.4, n 4.6 and n 4.7 we have used isomorphisms of the nilpotent algebras to cast the individual entries in a convenient form, yielding simple solutions for the same choice of complex structure on the base T^4. The same isomorphism applied to n 4.5 gives the algebra $(0,0,0,0,2\times(14-13)+23-24, 23-13+2\times(24-14))$.

with $J_{\mathcal{B}} = \frac{i}{2}(\chi^1 \wedge \overline{\chi}^{\bar{1}} + \chi^2 \wedge \overline{\chi}^{\bar{2}})$, $\omega_{\mathcal{B}} = \chi^1 \wedge \chi^2$ and $\Theta = \mathrm{d}z + \alpha$.

Note that in order to obtain a geometric background, we need to perform the twist along isometries. As in standard T-duality, this implies a smearing in the fiber directions, especially for the warp factor. Then we expect to have O5 planes in the directions 56.

To determine the connection, as well as the other fields in the solution, we can require that the transformed background satisfies the supersymmetry conditions (4.4.1) for O5 compactifications with type 3 - type 0 pure spinors [29]

$$\begin{aligned}
&e^{\phi'} = g_s e^{2A'} \\
&\mathrm{d}(e^{A'}\Omega') = 0 \\
&\mathrm{d}(J')^2 = 0 \\
&\mathrm{d}(e^{2A'}J') = g_s e^{4A'} * F_3' \\
&H = 0\,.
\end{aligned} \tag{4.4.15}$$

Also, the O5 projection sets $\theta_+ = 0$ and we choose again $\theta_- = 0$, hence $\theta_c^- = 0$ and $\theta_c^+ = -\pi/2$.

Equivalently, we can use the constraints worked out in (4.4.4). It is straightforward to verify that from the equation for the real part of Ψ'_+, it follows that indeed $e^{\phi'} = g_s e^{2A'}$ and

$$g'_{z\bar{z}} = e^{2A'} \qquad \begin{aligned} F \wedge J_{\mathcal{B}} = 0\,, \\ \overline{F} \wedge J_{\mathcal{B}} = 0\,, \end{aligned} \tag{4.4.16}$$

where we introduced the curvature F as $\mathrm{d}\Theta = \pi^* F$. Similarly, the imaginary part of Ψ'_+ can be used to define the RR three-form as in (4.4.15) (see also (4.4.5)). Finally, the equation for Ψ'_- sets $A' = A$ and

$$F \wedge \omega_{\mathcal{B}} = 0\,. \tag{4.4.17}$$

Using the form (4.4.16) for the new metric on the fiber and the fact that the warp factor does not change, we can write the metric on M as

$$\mathrm{d}s_6^2 = e^{-2A} \sum_{\tilde{m}=1}^{4} (\mathrm{d}x^{\tilde{m}})^2 + e^{2A} \sum_{I=1,2} (\mathrm{d}x^I + A^I)^2\,, \tag{4.4.18}$$

which is indeed what one expects for O5 compactifications. As a transformation on the generalized vielbein, (4.3.7), the twist acts as

$$A_{\mathcal{B}} = \mathbb{I}_4\,, \quad A_{\mathcal{F}} = \mathbb{I}_2 \times e^{2A'}\,, \quad A_{\mathcal{C}}{}^{z}{}_{\tilde{m}} = e^{2A'}\alpha_{\tilde{m}}\,, \quad A_{\mathcal{C}}{}^{\bar{z}}{}_{\tilde{m}} = e^{2A'}\overline{\alpha}_{\tilde{m}}\,, \tag{4.4.19}$$

and we can check the dilaton is transformed as expected.

Let us go back to the form of the constraints on the curvature F. From (4.4.16) and (4.4.17), we see that demanding that the twist preserves supersymmetry is equivalent to the requirement that F does not have a purely anti-holomorphic part and its contraction with the Kähler form on $\mathcal{B}$ vanishes

$$F = F^{2,0} + F_-^{1,1}\,. \tag{4.4.20}$$

Using the diagonal metric on T^4 associated to the Kähler form $J_{\mathcal{B}}$, it is convenient to define an orthogonal set of two-forms

$$\begin{aligned}
j_\pm^1 &= e^1 \wedge e^2 \pm e^3 \wedge e^4\,, \\
j_\pm^2 &= e^1 \wedge e^3 \mp e^2 \wedge e^4\,, \\
j_\pm^3 &= e^1 \wedge e^4 \pm e^2 \wedge e^3\,,
\end{aligned} \tag{4.4.21}$$

such that $j_\pm^i = \pm * j_\pm^i$ (for $i = 1,2,3$) and $j_\pm^i \wedge j_\pm^j = \pm\frac{1}{2}\delta^{ij}\,\mathrm{vol}(T^4)$. Then $J_{\mathcal{B}} = j_+^1$ and $\omega_{\mathcal{B}} = j_+^2 + i j_+^3$. The decomposition (4.4.20) becomes

$$F = f_+(j_+^2 + i j_+^3) + f_i\, j_-^i \tag{4.4.22}$$

for a set of complex f_+, f_i. It is not hard to verify now that $f_+ = 1, f_i = (0,0,0)$ for n 4.7, $f_+ = 1, f_i = (0,1,0)$ for n 4.4, $f_+ = \frac{1}{2}, f_i = (0, \frac{1}{2}, \frac{i}{2})$ for n 4.6, and $f_+ = -\frac{1+3i}{2}, f_i = (0, \frac{i-3}{2}, \frac{1-3i}{2})$ for n 4.5. Hence the curvatures for these four cases satisfy the conditions needed to preserve supersymmetry.

When F is purely imaginary (real) we get a special case of a single non-trivial circle fibration. Indeed after setting to zero f_+ and the real part of f_i, the algebra n 4.6 becomes (0,0,0,0,0,$\frac{1}{2}\times$ (14-23)), which is isomorphic to n 5.1. Similarly, either by setting to zero f_+ and the imaginary part of f_i in n 4.6 (modulo the factor $\frac{1}{2}$), or by simply setting to zero f_+ in n 4.4, one gets a nilpotent algebra (0,0,0,0,13+24,0) which is again isomorphic to n 5.1. For n 4.5 one of the two $U(1)$'s can also be chosen trivial; the non-trivial fibration will be in a direction that is a linear combination of x^5 and x^6.

So the twist transformation enables us to recover all but one type IIB solutions with $SU(3)$ structure mentioned in section 3.3. Out of the constraints, we understand better why these manifolds only do provide solutions. We recall again that all the type C solutions on these nilmanifolds can also be obtained by ordinary T-duality from a type B solution with a specific choice of NSNS flux. We are now going to study the remaining solution, the one which has not been recovered yet. On the contrary to the other solutions, this one does not admit such a type B T-dual solution.

Iterating the twist for a second fibration

The list of IIB $SU(3)$ structure solutions with O5/D5 sources on nilmanifolds given in section 3.3 includes only one case which is not related by a sequence of T-dualities to flux compactifications on straight T^6. The existence of such isolated solution is somewhat puzzling, and, as we shall see, it is related to the rest of nilmanifold compactifications by the twist transformation.

The manifolds n 4.3 and n 4.6 have trivial S^1 factors. These can be twisted as well, moving us in the table of nilmanifolds into the domain of lower b_1. In particular n 4.6 has the form $M = S^1 \times N_5$ where N_5 is a T^2 fibration over T^3. The second cohomology of N_5 is non-trivial ($b_2(N_5) = 6$) and hence it can support non-trivial $U(1)$ bundles. A priori there can be up to six different ways of constructing a $U(1)$ fibration and there are several topologically distinct ways to produce a manifold with $b_1(M)$=3 out of n 4.6. However we will see that one of them is singled out by supersymmetry.

In the previous section it was shown that n 4.6 yields a type IIB solution with O5/D5 sources. We want to further twist the remaining $U(1)$ bundle[11] without changing the type of sources. This requires taking a real twist of the S^1 factor. From the n 4.6 algebra (0,0,0,0,13,14) it is not hard to see that the S^1 corresponds to the direction 2, and hence the twisting amounts to sending $\mathrm{d}\tilde{e}^2 = 0$ to $\mathrm{d}\tilde{e}^2 = F$ where $F \in H^2(N_5)$. The algebra[12] becomes (0,F,0,0,13,14). The form of the F is again fixed by imposing that the supersymmetry equations (4.4.15) continue to hold. This yields the conditions

$$\begin{aligned} F \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) &= 0\,, \\ F \wedge (e^1 \wedge e^3 \wedge e^4 + e^1 \wedge e^5 \wedge e^6) &= 0\,, \end{aligned} \tag{4.4.23}$$

which are solved by $F = \tilde{e}^3 \wedge \tilde{e}^5 + \tilde{e}^4 \wedge \tilde{e}^6$, where we set $\tilde{e}^i = e^A e^i$ for $i = 1, \ldots, 4$ and $\tilde{e}^i = e^{-A} e^i$ for $i = 5, 6$. The corresponding algebra is (0,35+46,0,0,13,14), which is indeed isomorphic to n 3.14, (0,0,0,12,23,14 - 35). In [29] it was shown that n 3.14 corresponds to the only solution involving nilmanifolds that was not obtained by T-duality from compactifications on T^6 with fluxes. Our twist transformation does connect it to the rest of the nilmanifold solutions family.

A typical feature of such non T-dual solutions is that they involve non-localized intersecting sources, in this case two O5 planes. It is easy to see that our twist leads to the same result. Indeed, the Bianchi identity for the F_3 flux

$$g_s \mathrm{d}F_3 = 2i\partial\bar{\partial}(e^{-2A} J) = \delta(\text{source})\ , \tag{4.4.24}$$

[11] As explained in appendix C.1.1, to obtain a nilmanifold with two iterated fibrations, one has indeed to perform two consecutive twists, as we do here.

[12] It is easy to check that F is a linear combination of $e^1 \wedge e^5$, $e^1 \wedge e^6$, $e^3 \wedge e^5$, $e^3 \wedge e^4$, $e^4 \wedge e^6$ and $e^3 \wedge e^6 + e^4 \wedge e^5$.

with the Kähler form

$$J = e^{-2A}(\tilde{e}^1 \wedge \tilde{e}^2 + \tilde{e}^3 \wedge \tilde{e}^4) + e^{2A}\tilde{e}^5 \wedge \tilde{e}^6 \,, \tag{4.4.25}$$

becomes

$$\begin{aligned} g_s \mathrm{d}F_3 &= 2[\nabla(e^{-2A}) - e^{2A}]\tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^4 + 2e^{-2A}\,\tilde{e}^3 \wedge \tilde{e}^4 \wedge \tilde{e}^5 \wedge \tilde{e}^6 \\ &\quad + \mathrm{d}(e^{-2A})(\tilde{e}^2 \wedge \tilde{e}^4 \wedge \tilde{e}^6 + \tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^5) + \mathrm{d}(e^{2A})(\tilde{e}^2 \wedge \tilde{e}^4 \wedge \tilde{e}^5 - \tilde{e}^2 \wedge \tilde{e}^3 \wedge \tilde{e}^6)\,. \end{aligned} \tag{4.4.26}$$

In order to be consistent with the calibration conditions for the sources (see section 3.3 and the appendix B.3), the last line should vanish. Had we assumed that ∂_5, ∂_6 and ∂_2 are all honest isometries, this would set A to a constant, thus giving the unsurprising result that due to the intersection the sources are smeared[13]. There is however the possibility of keeping the x^2 dependence in the warp factor and still have a consistent (and partially localized) solution [43, 44]. This possibility assumes that the last twist (in direction 2) did not really require an isometry but a circle action. This also suggests a possible generalization of our procedure, but we shall not pursue this further.

4.4.3 Generating solutions on solvmanifolds

In the previous section, we showed that the twist transformation could relate the known $SU(3)$ structure solutions on nilmanifolds to solutions on T^6, being T-duals or not. Let us focus here on solvmanifolds. Few $SU(3)$ structure solutions are known on solvmanifolds, as mentioned in section 3.3. These solutions all have two smeared intersecting sources, and are not T-dual to a T^6 solution, but as in the previous section, they can be related to a T^6 solution by a twist transformation.

Here, we will rather use the transformation to generate a new solution. As shown in section 4.2, the solvmanifold $\mathfrak{g}_{5.17}^{0,0,\pm 1} \times S^1$ is related by the twist to the more general manifold $\mathfrak{g}_{5.17}^{p,-p,\pm 1} \times S^1$. It is then natural to ask what is the effect of twisting the type IIA solution in [29]. At the end of this section, we present additionally a new localized solution on s 2.5 with one $O6$ source.

We take as starting point Model 3 of [29]. This is an $SU(3)$ structure solution with smeared $D6$-branes and $O6$-planes in the directions (146) and (236). For an $SU(3)$ structure, the two pure spinors $\Psi_\pm$ are given in (2.4.13). The phase in Ψ_+ is, in general, determined by the orientifold projection. For O6 planes θ_+ is actually free and we set it to zero. We take for the $SU(3)$ forms

$$\Omega = \sqrt{t_1 t_2 t_3}\,\chi^1 \wedge \chi^2 \wedge \chi^3 \qquad\qquad J = \frac{i}{2}\sum_k t_k \chi^k \wedge \overline{\chi}^k \,, \tag{4.4.27}$$

with complex structure[14]

$$\begin{aligned} \chi^1 &= e^1 + i\,\lambda\frac{\tau_3}{\tau_4}\,e^2 \,, \\ \chi^2 &= \tau_3\,e^3 + i\tau_4\,e^4 \,, \\ \chi^3 &= e^5 - i\tau_6\,e^6 \,. \end{aligned} \tag{4.4.29}$$

For simplicity, we introduce $\lambda = \frac{t_2\tau_4^2}{t_1}$. e^m are globally defined one-forms, obtained as in (4.2.1)

$$e^m = (A_2)^m{}_n \mathrm{d}x^n \,, \tag{4.4.30}$$

[13]Notice that while keeping the transformation real ensures that there is no change in the type of solution and hence the sources (both the solution involving n 4.6 and the one on n 3.14 are of type C and have O5/D5 sources), relative orientations of individual sources can change.

[14]Ω and J are normalised as

$$\frac{4}{3}J^3 = i\Omega \wedge \overline{\Omega} = -8\ \mathrm{vol}_{(6)} = -8\ \sqrt{|g|}\ e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6 \tag{4.4.28}$$

where $\mathrm{vol}_{(6)}$ is the internal volume form.

with A_2 given by (4.2.19). With this choice the metric in the e^m basis is diagonal

$$g = \mathrm{diag}\left(t_1, \lambda\, t_2\, \tau_3^2, t_2\, \tau_3^2, \lambda t_1, t_3, t_3\tau_6^2\right) . \tag{4.4.31}$$

Positivity of the volume and definite-positiveness of the metric imposes the following constraints on the complex structure and Kähler moduli

$$\tau_6 > 0\ ,\ t_1,\ t_2,\ t_3 > 0\,. \tag{4.4.32}$$

Due to the presence of intersecting sources, the warp factor is set to one and the dilaton to a constant. By splitting the pure spinor equations (2.4.18), (2.4.19), and (2.4.20) into forms of fixed degree, it is easy to verify that supersymmetry implies

$$\mathrm{d}(\mathrm{Im}\,\Omega) = 0\,, \tag{4.4.33}$$
$$\mathrm{d}J = 0\,, \tag{4.4.34}$$
$$\mathrm{d}(\mathrm{Re}\,\Omega) = g_s * F_2\,, \tag{4.4.35}$$
$$F_6 = F_4 = F_0 = H = 0\,. \tag{4.4.36}$$

The only non-zero RR flux reads

$$g_s F_2 = \frac{\sqrt{\lambda}\,(q_1 t_1 - q_2 t_2 \tau_3^2)}{\sqrt{t_3}}(e^3\wedge e^4 - e^1\wedge e^2)\,, \tag{4.4.37}$$

and it is straightforward to check that its Bianchi identity is satisfied.

Given the solution above, we want to use the twist action to produce solutions, still with $O6$-planes and $D6$-branes, on $\mathfrak{g}_{5.17}^{p,-p,\pm1}\times S^1$. The manifolds $\mathfrak{g}_{5.17}^{p,-p,\pm1}\times S^1$ and $\mathfrak{g}_{5.17}^{0,0,\pm1}\times S^1$ are related by the twist matrix A_1 in (4.2.19), whose adjoint matrix is

$$ad_{\partial_5}(\mathfrak{n})|_p = \begin{pmatrix} a_{12} & \\ & a_{34}\end{pmatrix} \qquad\qquad a_{12} = a_{34}^T = \begin{pmatrix} 0 & pq_1 \\ pq_2 & 0\end{pmatrix} . \tag{4.4.38}$$

The sixth direction being a trivial circle, we identify $t = x^5$. Then the twist action on pure spinors,

$$\Psi_\pm \mapsto \Psi'_\pm = O\cdot\Psi_\pm\,, \tag{4.4.39}$$

can be rewritten as (see (4.3.25))

$$\begin{aligned} O &= e^{-px^5(q_2 e^1\wedge\iota_2 + q_1 e^2\wedge\iota_1)}\, e^{-px^5(q_1 e^3\wedge\iota_4 + q_2 e^4\wedge\iota_3)} \\ &= O_{12}O_{34}\,, \end{aligned} \tag{4.4.40}$$

with

$$\begin{aligned} O_{12} &= \mathbb{I} + [\cosh(p\sqrt{q_1q_2}x^5) - 1](e^1\wedge\iota_1 + e^2\wedge\iota_2 + 2e^1\wedge e^2\wedge\iota_1\wedge\iota_2) \\ &\quad -\frac{1}{\sqrt{q_1q_2}}\sinh(p\sqrt{q_1q_2}x^5)(q_2e^1\wedge\iota_2 + q_1e^2\wedge\iota_1)\,, \end{aligned} \tag{4.4.41}$$

$$\begin{aligned} O_{34} &= \mathbb{I} + [\cosh(p\sqrt{q_1q_2}x^5) - 1](e^3\wedge\iota_3 + e^4\wedge\iota_4 + 2e^3\wedge e^4\wedge\iota_3\wedge\iota_4) \\ &\quad -\frac{1}{\sqrt{q_1q_2}}\sinh(p\sqrt{q_1q_2}x^5)(q_1e^3\wedge\iota_4 + q_2e^4\wedge\iota_3)\,. \end{aligned} \tag{4.4.42}$$

Note that unimodularity of the algebra implies $\det(A) = 1$. In comparison to (2.3.44), here we do not introduce a phase in the twist operator, since we do not modify the nature of the fluxes and sources. So the only component of the twist transformation that we use here is what we called the topology change.

It is straightforward to check that the transformed pure spinors and the $SU(3)$ structure forms have formally the same expression as in (4.4.27) - (4.4.29) but with the one-forms e^m now given by

$$e^m = (A_1 A_2)^m{}_n \mathrm{d}x^n \,. \tag{4.4.43}$$

Also the metric, which is completely specified by the pure spinors, has the same form as for the initial solution, but in the new e^m basis

$$g = \mathrm{diag}\left(t_1, \lambda t_2\tau_3^2, t_2\tau_3^2, \lambda t_1, t_3, t_3\tau_6^2\right) \,. \tag{4.4.44}$$

In order for the twist transformation to produce new solutions, the transformed pure spinors should again satisfy the supersymmetry equations (4.4.3), or equivalently the constraints (4.4.4). The conditions

$$H' = 0 \qquad\qquad \mathrm{d}J' = 0 \tag{4.4.45}$$

are automatically satisfied, so that the first two equations in (4.4.3) reduce to[15]

$$0 = \mathrm{d}(\mathrm{Im}\,\Omega') = -p(\lambda-1)\ \tau_3\tau_6\sqrt{t_1t_2t_3}\ (q_2\ e^1\wedge e^4\wedge e^5 + q_1\ e^2\wedge e^3\wedge e^5)\wedge e^6 \,. \tag{4.4.47}$$

From this we see that, beside the solution with $p = 0$, other supersymmetric solutions exist with $p \neq 0$ provided $\lambda = 1$.

The last SUSY equation in (4.4.3) defines the transformed RR field as in (4.4.5). Since the twist operator does not change the degree of forms, it follows from (4.4.5) that no new RR fluxes have been generated

$$F_0 = F_4 = F_6 = 0 \,, \tag{4.4.48}$$

and (we have already set $\lambda = 1$)

$$g_sF_2 = \frac{q_1t_1 - q_2t_2\tau_3^2}{\sqrt{t_3}}(e^3\wedge e^4 - e^1\wedge e^2) + \frac{p(q_1t_1 + q_2t_2\tau_3^2)}{\sqrt{t_3}}(e^2\wedge e^4 + e^1\wedge e^3) \,. \tag{4.4.49}$$

The Bianchi identity for F_2 is satisfied

$$g_s\mathrm{d}F_2 = c_1v^1 + c_2v^2 \ , \tag{4.4.50}$$

with $v^1 = t_1\sqrt{t_3}\ e^1\wedge e^4\wedge e^5$ and $v^2 = t_2\tau_3^2\sqrt{t_3}\ e^2\wedge e^3\wedge e^5$ being the covolumes of the sources in (236) and (146) (see (3.3.5)). The sign of the charges

$$\begin{aligned} c_1 &= \frac{2q_2}{t_3t_1}\left[t_1q_1(1-p^2) - (1+p^2)t_2q_2\tau_3^2\right] \\ c_2 &= \frac{2q_1}{t_3t_2\tau_3^2}\left[\tau_3^2t_2q_2(1-p^2) - (1+p^2)t_1q_1\right] \end{aligned} \tag{4.4.51}$$

depends on the parameters, but the sum of the two charges is clearly negative. This guarantees that the transformed background with $p \neq 0$ and $\lambda = 1$ is indeed a solution of the full set of ten-dimensional equations of motion. In the next chapter we will use the non-supersymmetric version of this background, with $\lambda \neq 1$, as starting point for our search for de Sitter solution.

[15] Note that a slightly more general solution given by $\chi^1 = e^1 + i\left(\frac{\tau_3}{\tau_4}\lambda\ e^2 - \frac{\tau_2}{\tau_4}\ e^3\right)$, $\chi^2 = \tau_2\ e^2 + \tau_3\ e^3 + i\tau_4\ e^4$ and the same χ^3 leads to the same $\mathrm{d}(\mathrm{Im}\,\Omega')$ and to

$$\mathrm{d}(J') = -p(\lambda-1)\ \tau_2\sqrt{\frac{t_1t_2}{\lambda}}\ (q_2\ e^1\wedge e^4\wedge e^5 + q_1\ e^2\wedge e^3\wedge e^5) \,. \tag{4.4.46}$$

A supersymmetric solution, requiring $\mathrm{d}(\mathrm{Im}\Omega) = \mathrm{d}J = 0$, needs $\lambda = 1$. For $\tau_2 = 0$ we can have non-supersymmetric solutions with a closed J'.

For completeness, it is worth to give the explicit form of the torsion classes for our solution. For $SU(3)$ structure solutions, the deviation from closure of the holomorphic three-form and the Kähler form, is parametrized by five torsion classes

$$\begin{aligned} \mathrm{d}J &= \frac{3}{2}\,\mathrm{Im}(\bar{W}_1\Omega) + W_4 \wedge J + W_3 \\ \mathrm{d}\Omega &= W_1 J^2 + W_2 \wedge J + \bar{W}_5 \wedge \Omega\,, \end{aligned} \tag{4.4.52}$$

where W_1 is a complex scalar, W_2 is a complex primitive $(1,1)$ form, W_3 is a real primitive $(2,1)+(1,2)$ form, W_4 is a real vector and W_5 is a complex $(1,0)$ form. For the $SU(3)$ structure solution ($p \neq 0$, $\lambda \neq 1$, $\tau_2 \neq 0$) mentioned in footnote 15, we obtain

$$\begin{aligned} W_1 &= \frac{p\,\tau_2\,(A+B)(1-\lambda)}{6(\tau_2^2+\lambda\tau_3^2)\sqrt{t_1t_2t_3}} \\ W_2 &= \frac{1}{6(\tau_2^2+\lambda\tau_3^2)\sqrt{t_1t_2t_3}}\Big[-it_1\Big(p\tau_2\,(A+B)(\lambda+2)+3\lambda\tau_3(A-B)\Big)\chi^1\wedge\bar{\chi}^1 + \\ &\quad +3\sqrt{\lambda t_1t_2}\Big(\tau_2(B-A)+p\tau_3(\lambda A+B)\Big)\chi^1\wedge\bar{\chi}^2 - 3\sqrt{\lambda t_1t_2}\Big(\tau_2(B-A)+p\tau_3(A+\lambda B)\Big)\chi^2\wedge\bar{\chi}^1 + \\ &\quad +it_2\Big(p\tau_2(A+B)(1+2\lambda)+3\lambda\tau_3(A-B)\Big)\chi^2\wedge\bar{\chi}^2 - ip\tau_2t_3(A+B)(\lambda-1)\chi^3\wedge\bar{\chi}^3\Big] \\ W_3 &= \frac{ip\tau_2(\lambda-1)}{8(\tau_2^2+\lambda\tau_3^2)}\Big[(A+B)\chi^1\wedge\chi^2\wedge\bar{\chi}^3-(A+B)\chi^3\wedge\bar{\chi}^1\wedge\bar{\chi}^2 + \\ &\quad -(A-B)(\chi^1\wedge\chi^3\wedge\bar{\chi}^2-\chi^1\wedge\bar{\chi}^2\wedge\bar{\chi}^3+\chi^2\wedge\chi^3\wedge\bar{\chi}^1-\chi^2\wedge\bar{\chi}^1\wedge\bar{\chi}^3)\Big] \\ W_4 &= 0 \\ W_5 &= \frac{ip\sqrt{\lambda}\tau_3(A+B)(\lambda-1)}{4(\tau_2^2+\lambda\tau_3^2)\sqrt{t_1t_2}}\,\chi^3\,, \end{aligned} \tag{4.4.53}$$

with $A = q_1t_1$, $B = q_2t_2(\tau_3^2+\frac{\tau_2^2}{\lambda})$.

Solution with localized source, and the warping

The supersymmetric solution discussed previously is global, the warp factor and the dilaton being constant. It is an interesting question to see whether localized solutions also exist (see e.g. [65] for a recent discussion about the importance of warping). As discussed in section 3.3, the strategy for finding localized solutions used in [29] was first to look for a smeared solution at large volume and then localize it by scaling the vielbein, longitudinal and transverse with respect to the source, with e^A and e^{-A}, respectively. This procedure works provided only parallel sources are present. Unfortunately this is not the case for the supersymmetric solution we took as a departure point for our construction - the intersecting O6/D6 solution on $s\,2.5$.

It is however possible to find a completely localized solution on $s\,2.5$ with O6 planes. The solution has a simpler form in a basis where the algebra is $(25,-15,r45,-r35,0,0)$, $r^2=1$. In this basis the $O6$-plane is along the directions (345). The $SU(3)$ structure is constructed as in (4.4.27) with

$$\begin{aligned} \chi^1 &= e^{-A}e^1 + ie^A(\tau_3e^3+\tau_4e^4)\,, \\ \chi^2 &= e^{-A}e^2 + ie^A r(-\tau_4e^3+\tau_3e^4)\,, \\ \chi^3 &= e^Ae^5 + ie^{-A}r\tau_6e^6\,, \\ \tau_6 &> 0\,,\ t_1=t_2\,,\ t_3>0\,, \end{aligned} \tag{4.4.54}$$

where the non-trivial warp factor, e^{2A}, depends on x^1, x^2, x^6. The metric is diagonal

$$g = \mathrm{diag}\left(t_1 e^{-2A}, t_1 e^{-2A}, t_1(\tau_3^2+\tau_4^2)e^{2A}, t_1(\tau_3^2+\tau_4^2)e^{2A}, t_3 e^{2A}, t_3\tau_6^2 e^{-2A}\right) , \tag{4.4.55}$$

and the only non-zero flux is the RR two–form

$$g_s F_2 = -r\Big[\tau_6\sqrt{t_3}\partial_1(e^{-4A})\, \mathrm{d}x^2\wedge e^6 - \tau_6\sqrt{t_3}\partial_2(e^{-4A})\, \mathrm{d}x^1\wedge e^6 + \frac{1}{\tau_6}\sqrt{\frac{t_1^2}{t_3}}\partial_6(e^{-4A})\, \mathrm{d}x^1\wedge \mathrm{d}x^2\Big] . \tag{4.4.56}$$

Setting the parameters $t_1 = t_2$ in the Kähler form (4.4.27) allows to have a single source term in the F_2 Bianchi identity

$$g_s \mathrm{d}F_2 \sim e^{-A}\Delta(e^{-4A})e^1\wedge e^2\wedge e^6 , \tag{4.4.57}$$

where Δ is the laplacian with unwarped metric.

As $A \to 0$ this solution becomes fluxless ($s\,2.5$ can indeed support such solutions), hence it cannot be found following the strategy of localizing the large volume smeared solutions. Unfortunately this solution does not satisfy the twist to $p \neq 0$, (4.2.13), since for $p \neq 0$ the action of the involution of an $O6$-plane with a component along direction 5 is not compatible with the algebra.

4.4.4 A digression: twist and non-geometric backgrounds

We would like to come back to the question of the consistency of the twist transformation. As explained in section 4.2, the transformation is obstructed unless the matrix A is conjugated to an integer-valued matrix. In many cases, the twist can result in a topology change similar to what is achieved by T-duality. The latter also can be obstructed, and yet these obstructions do not stop us from performing the duality transformation. So what about the obstructed twist?

To keep things simple, let us consider again an almost abelian algebra and the gluing under $t \to t+t_0$. We should have in general

$$T_6 : \begin{cases} t \to t + t_0 \\ x^i \to \tilde{A}_M(-t_0)^i{}_j x^j \end{cases} \qquad i, j = 1, \dots, 5 , \tag{4.4.58}$$

where $\tilde{A}_M(-t_0)$ is necessarily an integer-valued matrix for $t_0 \neq 0$. In the case of compact almost abelian solvmanifolds, this matrix is given by (4.2.5). For the algebras that do not admit an action of a lattice, $\tilde{A}_M(-t_0)$ has nothing to do with the algebra. Then the one forms $e^m = A(t)^m{}_n \mathrm{d}x^n$ ($\mathrm{d}x^6 = \mathrm{d}t$) are defined only locally and have discontinuities under $t \to t + t_0$. These kinds of discontinuity are actually familiar from the situations when an obstructed T-duality is performed, and are commonly referred to as non-geometric backgrounds. One way to see this is to work on the generalized tangent bundle and use local $O(6) \times O(6)$ transformations (for six-dimensional internal manifolds) to bring the generalized vielbeine to the canonical lower diagonal form (2.3.23). In geometric backgrounds, this is a good transformation, while in the non-geometric case it involves non-single valued functions [27] (see also appendix C.2 for an illustration of this procedure, and the question of T-duals of solvmanifold solutions).

As an example, let us consider the manifold $\mathfrak{g}_{4.2}^{-p} \times T^2$, where the algebra $\mathfrak{g}_{4.2}^{-p}$ is given in appendix B.1.2. The corresponding group does not admit a lattice. For generic p this is very easy to see since the group is not unimodular. For $p = 2$, the group is unimodular but there still is no lattice. As explained in the Appendix B.1.2, in this case, the characteristic polynomial cannot have integer coefficients, and therefore there is an obstruction to the existence of a lattice.

If we now consider the algebra together with its dual, i.e. examine the existence of a lattice on the generalized tangent bundle E, we should study the 6×6 matrix $M(t) = \mathrm{diag}(\mu(t), \mu(-t)^T)$ instead of the matrix $\mu(t)$. One has

$$M(t) = \begin{pmatrix} e^{pt} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 & 0 & 0 \\ 0 & -te^{-t} & e^{-t} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-pt} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{t} & te^{t} \\ 0 & 0 & 0 & 0 & 0 & e^{t} \end{pmatrix} . \tag{4.4.59}$$

For $t_0 = \ln(\frac{3+\sqrt{5}}{2})$ and $p \in \mathbb{N}^*$, $M(t = t_0)$ is conjugated to an integer matrix, $P^{-1}M(t_0)P = N$, where N is an integer matrix (Theorem 8.3.2 in [37]):

$$P = \begin{pmatrix} 1 & 0 & 0 & \frac{18+8\sqrt{5}}{7+3\sqrt{5}} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{2(2+\sqrt{5})}{3+\sqrt{5}} \\ 0 & 0 & \ln(\frac{2}{3+\sqrt{5}}) & 0 & \frac{2(2+\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} & 0 \\ 1 & 0 & 0 & \frac{2}{3+\sqrt{5}} & 0 & 0 \\ 0 & 0 & \ln(\frac{2}{3+\sqrt{5}}) & 0 & -\frac{(1+\sqrt{5})\ln(\frac{3+\sqrt{5}}{2})}{3+\sqrt{5}} & 0 \\ 0 & -1 & 0 & 0 & 0 & \frac{1+\sqrt{5}}{3+\sqrt{5}} \end{pmatrix} , \tag{4.4.60}$$

$$N = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 & 1 & -1 \\ a_{41} & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.4.61}$$

The piece

$$N_4 = \begin{pmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}^p \tag{4.4.62}$$

comes from the entries e^{pt} and the result can be obtained[16] from (B.1.14). We see that on the generalized tangent bundle the basic obstruction to the existence of a lattice is easily removed. Moreover it is not hard to see that, due to putting together the algebra and its dual, even the requirement of unimodularity can be dropped.

On the generalized tangent bundle we can therefore obtain a lattice. For non-geometry, one may ask for more: the integer matrix N being in $O(3,3)$. This question can be decomposed into $N_4 \in O(1,1)$ and the 4×4 integer matrix in $O(2,2)$. Actually, the latter is true[17]. But $N_4 \notin O(1,1)$. Moreover, one can prove that $\mathrm{diag}(e^{pt}, e^{-pt})$ can only be conjugated to an integer $O(1,1)$ matrix for $t = 0$. Indeed, the eigenvalues of an integer $O(1,1)$ matrices are ±1, and those are not changed by conjugation.

This is reminiscent of the twist construction of the IIB background $n\,3.14$ discussed in section 4.4.2. The internal manifold is a circle fibration over a five manifold M_5, which itself is a bundle with a two-torus fiber, but the only obvious duality seen there is the $O(2,2)$ associated with the two-torus. The solution on $M_5 \times S^1$ is obtained from IIB solution on T^6 with a self-dual three-form flux, but not $n\,3.14$ itself [29].

By taking $p = 0$ in (4.4.59), we obtain a different topology. In $M(t)$ the corresponding direction becomes trivial, and we can forget about it. Up to an $O(1,1)$ action, the non-trivial part of $M(t)$

[16] Another possible conjugation is given in (3.2.18). The other part of N, the 4×4 integer matrix, can also be different, see the change of basis in Proposition 7.2.9 in [37].

[17] Note it is not true for the one given in Proposition 7.2.9 of [37].

can still be thought of as corresponding to the algebra on $T(\varepsilon_{1,1}) \oplus T^*(\varepsilon_{1,1})$. Indeed, $\varepsilon_{1,1}$ has two *local* isometries, and T-duality (the $O(1,1)$ in question) with respect to any of them will yields a non-geometric background. This can be inferred by simply noticing that the result of the duality in (any direction) is not unimodular; more detailed discussion of T-duality on $\varepsilon_{1,1}$ can be found in appendix C.2.

We would like to stress that, in order to apply the twist transformation to construct non-geometric backgrounds, a better understanding of the orientifold planes in GCG is clearly needed. However, the possibility of using solvable algebras in order to describe (some of) these is interesting.

4.5 Heterotic transformations

In this section we will apply the twist transformation to the heterotic string. Heterotic string provides the first examples where compactifications with non-trivial NSNS fluxes have been studied in full detail [18, 19]. We shall consider here the twist transformation on non-trivial flux backgrounds preserving at least $\mathcal{N}=1$ supersymmetry. The internal manifold will always be locally a product of $K3$ and T^2. As discussed in [61, 62] a chain of dualities can relate a solution involving $K3 \times T^2$ to one where the internal space is given by a non-trivial T^2 fibration over $K3$ (with connection). It is natural to ask whether they could be related by an $O(6,6)$ transformation of the type (4.3.7).

As we discussed in the section 2.3, the action of $O(6,6)$ is naturally implemented in the Generalized Complex Geometry framework. Such an approach is missing for the heterotic string, basically because of the absence of a good twisting of the exterior derivative. It is nevertheless possible to derive differential equations on pure spinors that capture completely the information contained in the supersymmetry variations. This is all we need to act with the $O(6,6)$ transformation (4.3.7). In this section we will derive the equations for the pure spinors in the heterotic string and use them to build the $O(6,6)$ transformation connecting the $SU(3)$ structure solutions of [61].

4.5.1 $\mathcal{N}=1$ supersymmetry conditions

Before writing the pure spinor equations for $\mathcal{N}=1$ compactifications in the heterotic case, we will briefly recall the conditions for $\mathcal{N}=1$ supersymmetry [18, 19].

The supersymmetry equation for the heterotic case can be written[18]

$$\begin{aligned}
\delta\psi_M &= (D_M - \frac{1}{4}H_M)\epsilon = 0\,, \\
\delta\lambda &= (\partial\!\!\!/\phi - \frac{1}{2}\,H\!\!\!\!/\,)\epsilon = 0\,, \\
\delta\chi &= 2\,\mathcal{F}\!\!\!\!/\,\epsilon = 0\,,
\end{aligned} \tag{4.5.1}$$

where ϵ is a positive chirality ten-dimensional spinor. $\mathcal{F}$ is the gauge field strength taken to be hermitian[19], i.e. defined with the following covariant derivative on the gauge connection $\mathcal{A}$

$$\mathcal{F} = (\mathrm{d} - i\mathcal{A}\wedge)\mathcal{A}\ . \tag{4.5.2}$$

The conditions that $\mathcal{N}=1$ supersymmetry imposes on compactifications to a four-dimensional maximally symmetric space and non-trivial NSNS flux were derived in [18]. If we write the ten-dimensional string frame metric as in type II, (2.2.1),

$$\mathrm{d}s^2 = e^{2A}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + g_{mn}(y)\mathrm{d}y^m\mathrm{d}y^n\,, \tag{4.5.3}$$

[18] These conventions are the same as in type II [4] with the RR fluxes set to zero. Note that these are related to the conventions of [58] via $H \to -H$.

[19] Following conventions of [58], we can develop the gauge quantities in terms of hermitian generators λ^a in the vector representation of $SO(32)$, and we use the normalisation condition $tr(\lambda^a\lambda^b) = 2\delta^{ab}$.

then the warp factor must be zero $A = 0$ and the four-dimensional metric Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} \, . \tag{4.5.4}$$

The internal manifold must be complex. The holomorphic three-form Ω satisfies

$$\mathrm{d}(e^{-2\phi}\Omega) = 0 \, . \tag{4.5.5}$$

In terms of the complex structure I defined by Ω, the Kähler form is $J_{mn} = I_m{}^p g_{pn}$ and satisfies

$$\mathrm{d}J = i(H^{1,2} - H^{2,1}) \Leftrightarrow H = i(\partial - \overline{\partial})J \, , \tag{4.5.6}$$
$$\mathrm{d}(e^{-2\phi} J \wedge J) = 0 \, . \tag{4.5.7}$$

The NSNS three-form has only components $(2,1)$ and $(1,2)$ with respect to the complex structure $I_n{}^m$

$$H = H_0^{2,1} + H_0^{1,2} + (H^{1,0} + H^{0,1}) \wedge J \, , \tag{4.5.8}$$

where the subindex 0 denotes the primitive part of H. The gauge field strength $\mathcal{F}$ must satisfy the six-dimensional hermitian Yang-Mills equation, i.e. must be of type $(1,1)$ and primitive

$$\mathcal{F} \llcorner J = 0 \, , \tag{4.5.9}$$
$$\mathcal{F}_{ij} = \mathcal{F}_{\overline{ij}} = 0 \, , \tag{4.5.10}$$

where the second equation is given in holomorphic and anti-holomorphic indices.

These are the necessary conditions imposed by supersymmetry. The equations of motion are satisfied provided the Bianchi identity holds:

$$H = \mathrm{d}B - \frac{\alpha'}{4} tr \left(\mathcal{A} \wedge \mathrm{d}\mathcal{A} - i\frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) + \frac{\alpha'}{4} \omega_3(M) \, , \tag{4.5.11}$$

where $\mathcal{A}$ is the gauge connection and $\omega_3(M)$ the Lorentz Chern-Simons term [16]. It is easier to check the anomaly cancellation condition

$$\mathrm{d}H = 2i\overline{\partial}\partial J = \frac{\alpha'}{4} [tr(\mathcal{R} \wedge \mathcal{R}) - tr(\mathcal{F} \wedge \mathcal{F})] \, . \tag{4.5.12}$$

4.5.2 Pure spinor equations for heterotic compactifications

In the four plus six-dimensional splitting, the supersymmetry parameter ϵ corresponds, for $\mathcal{N} = 1$ supersymmetry, to a single six-dimensional chiral spinor η_+

$$\epsilon = \zeta_+ \otimes \eta_+ + \zeta_- \otimes \eta_- \, , \tag{4.5.13}$$

where ζ_+ is, as always, a four-dimensional Weyl spinor of positive chirality ($\zeta_- = (\zeta_+)^*$) and $\eta_- = (\eta_+)^*$. The spinor η_+ can be seen as defining an $SU(3)$ structure on M (and indeed the supersymmetry conditions can be rephrased in terms of conditions on the torsion classes of an $SU(3)$ structure manifold). Then a natural choice for the pure spinors is

$$\begin{aligned} \Psi_+ &= 8\, e^{-\phi} \eta_+ \otimes \eta_+^\dagger = e^{-\phi}\, e^{-iJ} \, , \\ \Psi_- &= 8\, e^{-\phi} \eta_+ \otimes \eta_-^\dagger = -ie^{-\phi}\Omega \, . \end{aligned} \tag{4.5.14}$$

We have used the same letter as in (2.4.15) and (2.4.16) for the fermion bilinears (4.5.14), and we will still call them pure spinors. However it should be kept in mind that they are not defined on the generalized tangent bundle E but on $TM \oplus T^*M$ (e^{-B} is missing). Using (4.5.1) and (4.5.13), one can obtain the supersymmetry conditions on the pure spinors [4, 29]

$$\mathrm{d}\,(\Psi_\pm) = H \bullet \Psi_\pm \;, \tag{4.5.15}$$

with

$$H \bullet \Psi_\pm = \frac{1}{4} H_{mnp} (\mathrm{d}x^m \wedge \mathrm{d}x^n \wedge \iota^p - \frac{1}{3}\iota^m \iota^n \iota^p)\Psi_\pm \;. \tag{4.5.16}$$

Even though (4.5.15) captures all the information contained in supersymmetry variations, there are two problems with the action of the ($\mathrm{d} - H\bullet$) operator: it is not a differential, and it is hard to interpret its action on pure spinors as a twisting. There is a partial resolution to the former problem. The Ψ_- equation yields that H is indeed only of (1,2) + (2,1) type as given in (4.5.8), and

$$\mathrm{d}\,(\Psi_-) = i H^{0,1} \wedge \Psi_- \;, \tag{4.5.17}$$

from which we conclude that the internal manifold is complex. We can now use the integrability of the complex structure (4.5.17) to rewrite (4.5.15) in terms of a differential

$$\mathrm{d}\,(\Psi_\pm) = \pm \Big[(H^{1,2} - H^{2,1}) - i(H^{0,1} - H^{1,0}) \Big] \wedge \Psi_\pm = \pm \check{H} \wedge \Psi_\pm \,. \tag{4.5.18}$$

The equation (4.5.18) for Ψ_- agrees with (4.5.17). The decomposition of the Ψ_+ equation by the rank of the differential forms gives

- at degree 1

$$\mathrm{d}\phi = i(H^{0,1} - H^{1,0}) \,, \tag{4.5.19}$$

  using which we recover the correct scaling on Ω (4.5.5).

- at degree 3

$$\begin{aligned} \mathrm{d}J &= i(H^{1,2} - H^{2,1}) && (4.5.20) \\ &= i(H_0^{1,2} - H_0^{2,1}) + \mathrm{d}\phi \wedge J \,. && (4.5.21) \end{aligned}$$

  Eq. (4.5.20) is clearly (4.5.6). Wedging (4.5.21) with J, we recover the balanced metric condition (4.5.7). Finally recalling that $*H = i(H_0^{2,1} - H_0^{1,2} - H^{1,0} \wedge J + H^{0,1} \wedge J)$ we arrive at

$$*H = -e^{2\phi} \mathrm{d}(e^{-2\phi} J) \;. \tag{4.5.22}$$

- at degree 5, there is no new information.

We can now check that $\mathrm{d} \mp \check{H}\wedge$ is a differential. Since $\check{H}$ is made of odd forms, it squares to zero, and, due to (4.5.19) and (4.5.20), $\mathrm{d}\check{H} = 0$. Hence $(\mathrm{d} \mp \check{H}\wedge)^2 = 0$.

There stays however the problem that we cannot see the action of $\mathrm{d} \mp \check{H}\wedge$ as a result of a twisting on the pure spinor. This will not prevent us for using the twist transformation to relate different heterotic backgrounds. Essentially the idea is to consider a very special case of the transformation (2.3.44) which does not contain a B-transform nor changes the phase of the pure spinor (even if this amounts to stepping back somewhat from the GCG). In other words, we keep only the $GL(d)$ part of the general transformation (2.3.44) and we demand that

$$(\mathrm{d} \mp \check{H}'\wedge)(O_c\, \Psi_\pm) = 0 \,. \tag{4.5.23}$$

Two internal geometries M and M', defined by the pairs $\Psi_\pm$ and $\Psi'_\pm = O_c\, \Psi_\pm$, are related via twisting and satisfy the same type of $\check{H}$-twisted integrability conditions. The pair of manifolds connected this way may in general be topologically and geometrically distinct. Examples of such connections were constructed recently in [66]. Since there is no B-transform involved in the construction, we are not dealing here with the transformations of the generalized tangent bundle. In this sense the discussion of the heterotic string differs from the rest of this chapter.

4.5.3 SU(3) structure solutions

We shall return to the class of fibered metrics discussed earlier. Consider a six-dimensional internal space with a four-dimensional base $\mathcal{B}$ which is a conformal Calabi-Yau, and a T^2 fiber with holomorphic coordinate $z = \theta^1 + i\theta^2$. The metric and the $SU(3)$ structure on the internal space are in general given by

$$\begin{aligned} \mathrm{d}s^2 &= e^{2\phi}\mathrm{d}s^2_{\mathcal{B}} + \Theta\overline{\Theta}\,, \\ J &= e^{2\phi}\, J_{\mathcal{B}} + \frac{i}{2}\Theta\wedge\overline{\Theta} \\ \Omega &= e^{2\phi}\,\omega_{\mathcal{B}}\wedge\Theta \end{aligned} \tag{4.5.24}$$

where $\Theta = \mathrm{d}z + \alpha$ and α is a $(1,0)$ connection one-form. $J_{\mathcal{B}}$ is the CY Kähler form, $\omega_{\mathcal{B}}$ is the CY holomorphic two-form, and the dilaton ϕ depends only on the base coordinates. Furthermore, the curvature of the T^2 bundle $F = \mathrm{d}\alpha$ has to be primitive with respect to $J_{\mathcal{B}}$

$$F\wedge J_{\mathcal{B}} = 0\,, \qquad \text{and} \qquad F\wedge\omega_{\mathcal{B}} = 0\,. \tag{4.5.25}$$

A general solution to these constraints is of the form $F = F^{+}_{(2,0)} + F^{-}_{(1,1)} \in H^{2,+}(\mathcal{B}) \oplus H^{2,-}(\mathcal{B})$. Then, one can satisfy the local supersymmetry equations, provided the base $\mathcal{B}$ is a four-dimensional hyper-Kähler surface. Here, the equations (4.5.5) and (4.5.7) are automatically satisfied.

In [61], two $\mathcal{N} = 2$ solutions with $\mathcal{B} = K3$ and a non-zero H have been discussed. In the first solution (which we will denote by Solution 1), the internal manifold is the direct product $K3\times T^2$, i.e. $\alpha = 0$. The gauge bundle is reduced to the sum of $U(1)$ bundles, so $\mathcal{F}$ is a sum of (1,1) primitive two-forms on the base. Furthermore, in this solution, $B = 0$, so H receives only α' contributions. The dilaton is non-trivial and the condition (4.5.6) relates its derivatives to the gauge term.

The second solution (Solution 2) consists of a non-trivial T^2 fibration[20] over $K3$, so we have an $\alpha \neq 0$. Moreover $\mathcal{F} = 0$ and $B = \mathrm{Re}(\alpha\wedge \mathrm{d}\overline{z}) \neq 0$. The dilaton is non-trivial, and has the same value as in the previous solution. The curvature of the connection α is in general given by (4.4.20), and the solution would then be $\mathcal{N} = 1$. If F has only a (1,1) part as in [61], the solution[21] is $\mathcal{N} = 2$.

These two solutions were proven to be related by a transition [54, 55, 58, 61, 62]. Both solutions arise from M-theory compactifications on $K3\times K3$. A first step consists in reducing to type IIB solutions on an orientifold $(T^4/\mathbb{Z}_2)\times(T^2/\mathbb{Z}_2)$. This is achieved by taking the two $K3$ at the point in moduli space where they both are $T^4/\mathbb{Z}_2$ orbifold. Then one of the two $T^4/\mathbb{Z}_2$ is considered as a fibration of T^2 over $T^2/\mathbb{Z}_2$, and the area of the fiber is taken to zero. This yields a type IIB solution on $(T^4/\mathbb{Z}_2)\times(T^2/\mathbb{Z}_2)$ with four $D7$ and one $O7$ at each of the four fixed points of $T^2/\mathbb{Z}_2$. Then two T-dualities along $T^2/\mathbb{Z}_2$ give a dual type IIB solution on $(T^4/\mathbb{Z}_2)\times(T^2/\mathbb{Z}_2)$ with $D9$ and $O9$ at the dual points. The same solution can also be interpreted as a type I solution on $K3\times T^2$ where $K3$ is understood as $T^4/\mathbb{Z}_2$. Finally, doing an S-duality, one gets the heterotic $SO(32)$ solution on $K3\times T^2$ where $K3$ is again understood as $T^4/\mathbb{Z}_2$. The transition between the two heterotic solutions then corresponds to an exchange of the two $K3$, and of its $(1,1)$ two-forms, namely $\mathcal{F}$ and F. Note that M-theory on $K3\times K3$ can be dual to type IIA on $X_3\times S^1$ where X_3 is a CY three-fold. Then, the exchange of the two $K3$ corresponds to mirror symmetry for X_3 [67]. This exchange should not change the dilaton, which is therefore the same in the two solutions.

[20] The Betti numbers are $b_1(M) = 0$, $b_2(M) = 20$ and $b_3(M) = 42$. Note that the Euler number $\chi(M)$ vanishes, thus the manifold has a global $SU(2)$ structure.

[21] The $\mathcal{N} = 2$ supersymmetry is easy to see using the $SU(2)$ structure. There exists a second pair of compatible pure spinors which are of type 1-2, namely $\Psi_+ = e^{-\phi}\omega_{\mathcal{B}}\wedge\exp(\Theta\wedge\overline{\Theta}/2)$ and $\Psi_- = e^{-\phi}\Theta\wedge\exp(-iJ_{\mathcal{B}})$ (where we chose $\theta_+ = \frac{\pi}{2}$, $\theta_- = \pi$). Differently from the type 0-3 pair, now it is Ψ_- which is not closed. The closure of Ψ_+ imposes a stronger condition than (4.5.25) requiring that $\omega_{\mathcal{B}}\wedge F^I = 0$ (for $I = 1, 2$) hence restricting $F = F^{-}_{(1,1)} \in H^{2,-}(\mathcal{B})$.

We may connect these two solutions directly via (the special case of) our transformation (4.3.32). Since we have a background with only two commuting isometries, the twist takes the form

$$O_c = 1 + o = 1 + \alpha \wedge i_{\partial z} + \overline{\alpha} \wedge i_{\partial \overline{z}} + \alpha \wedge \overline{\alpha} \wedge i_{\partial \overline{z}}\, i_{\partial z} \,. \tag{4.5.26}$$

It has the effect of sending $\mathrm{d}z$ to $\Theta = \mathrm{d}z + \alpha$ in the forms defining the $SU(3)$ structure (4.5.24), and hence it relates the internal geometries of two solutions. Since the only change in the metric between the two solutions is the presence of a non-trivial connection, we did not assume any rescaling of the metric and thus we set $A_{\mathcal{B}} = \mathbb{I}_4$ and $A_{\mathcal{F}} = \mathbb{I}_2$ in (4.3.32). As a consequence the dilaton does not change, in agreement with the analysis of [61].

Thus, starting with Solution 1 we read off the H from the closure of the transformed pure spinor (note the similarity with the type II situation (4.4.5))

$$\begin{aligned} H &= i(\partial - \overline{\partial})J \; = i(\partial - \overline{\partial})(e^{2\phi}) \wedge \; J_{\mathcal{B}} - \frac{1}{2}(\partial - \overline{\partial})\left((\mathrm{d}z + \alpha) \wedge (\mathrm{d}\overline{z} + \overline{\alpha})\right) \\ &= i(\partial - \overline{\partial})(e^{2\phi}) \wedge \; J_{\mathcal{B}} - \frac{1}{2}(\partial - \overline{\partial})\left(\alpha \wedge \overline{\alpha}\right) + \mathrm{d}\left(\mathrm{Re}(\alpha \wedge \mathrm{d}\overline{z})\right) \,, \end{aligned} \tag{4.5.27}$$

where we used the anti-holomorphicity of α. The last term is the only closed part of H, and comparing to (4.5.11) we derive the B-field of Solution 2

$$B = \mathrm{Re}(\alpha \wedge \mathrm{d}\overline{z}) \,. \tag{4.5.28}$$

Furthermore,

$$\mathrm{d}H = -2i\partial\overline{\partial}(e^{2\phi}) \wedge J_{\mathcal{B}} + F \wedge \overline{F} \,. \tag{4.5.29}$$

We would like to stress once more that the two solutions were related using the transformation on the fermion bilinears (4.5.14). Differently from the pure spinors in type II solutions these do not contain an e^{-B} factor and we have not performed any B-transform in mapping the solutions; rather the B-field was read off as the closed part of H.

The global aspects of the solutions deserve some comments. Eq.(4.5.29) has the same structure as the tadpole condition for the O5/D5 solutions in type IIB. Notice that, in general, the first term in (4.5.29) yields δ-function contributions which are associated with the positions of branes and planes, while the second term, after being completed to a top-form by wedging with J, integrates over the six-manifold M to a positive number. The presence of these defects is what makes T^2 fibrations over $\mathcal{B} = T^4$ an admissible basis for the solutions in IIB. In heterotic string in the absence of good candidates for negative tension defects, we would like to assume a smooth dilaton; the second term is then cancelled by the α' contributions to (4.5.11). Crucially, when $\mathcal{B} = K3$, terms like $\int_M \partial\overline{\partial}(e^{2\phi}) \wedge J^2$ vanish for any smooth ϕ, while for $\mathcal{B} = T^4$, ϕ may be non-single valued and the integral gives a finite contribution to the tadpole. Indeed it is known that compactifications on smooth T^2 fibrations over T^4 are forbidden by the heterotic Bianchi identity [57, 58]. Starting from a heterotic compactification on T^6 and applying the transformation (4.3.7) with non-single valued coefficients (and hence the dilaton) may allow to circumvent the constraints imposed by the Bianchi identity. However such backgrounds will be non-geometric and we will not discuss them further.

We conclude this section by turning briefly to the transformation of the gauge field $\mathcal{F}$. The ordinary $O(2, 18)$ transformation on the Narain lattice can exchange the antiself-dual part of the curvature of the T^2 fibration with the $U(1)$ factors in the gauge bundle. This exchange is consistent both with supersymmetry and tadpole cancellation. As discussed in [68], a better understanding of this exchange, as well as the transformation of the α' terms of H, is achieved considering the pullback of H to the total space of the gauge bundle $\tilde{\pi}$: $\mathcal{P} \to M$,

$$\mathcal{H} = \tilde{\pi}^* H - \frac{\alpha'}{4} tr \mathcal{A} \wedge \mathcal{F} \,,$$

whose contraction with the isometry vectors ∂_z and $\partial_{\overline{z}}$ gives a closed two-form (which can be exchanged with the gauge $U(1)$ curvature terms). For our purposes, in order to capture the transformation of the α' terms, one possibility is to extend the $O(d,d)$ action to $O(d+16, d+16)$ transformations, and introduce new generalized vielbein incorporating the gauge connection. We discuss this possibility in appendix C.3.

Chapter 5

Supersymmetry breaking sources and de Sitter vacua

5.1 Introduction

Recently a lot of activity in string compactifications has concentrated on the search for de Sitter solutions. This renewed interest is motivated by recent cosmological data suggesting that we leave in an expanding universe characterized by a small but positive cosmological constant.

De Sitter solutions are much harder to find than Minkowski or Anti de Sitter. A first difficulty arises from the fact that de Sitter space-time is not compatible with supersymmetry. As we saw in chapters 3 and 4, the vanishing of the SUSY variation together with the Bianchi identities for the fluxes imply the full set of equations of motion. Then, supersymmetry provides a huge technical simplification in the search for solutions since it allows to replace second order equations with first order ones.

A second problem concerns the possibility of having a positive cosmological constant Λ. As we will discuss in more details, for purely supergravity backgrounds, having $\Lambda > 0$ requires a non-trivial fine tuning of the geometric parameters and fluxes of the solutions.

Finally, given a ten-dimensional solution, one should ask its four-dimensional reduction to be stable, meaning the extrema of the four-dimensional potential should be minima[1]. This last requirement is also difficult to satisfy, and up to date, no stable de Sitter solution including only classical ten-dimensional ingredients has been found.

In this chapter we are interested in de Sitter solutions of type IIA supergravity. In this context, several no-go theorems against the existence of de Sitter vacua and ways of circumventing them have been proposed [12, 69, 70, 71, 42, 72, 73, 74, 75]. As a result, de Sitter vacua require some necessary (but not sufficient) assumptions. First, one needs O-planes as for Minkowski compactifications [12]. In particular we will consider O6/D6 sources. In that case, the internal manifold must have negative curvature and a non-zero Roman mass must be turned on [12, 70, 42, 73]. Another possibility is to allow for non-geometric fluxes, but we will not pursue this approach here.

Therefore, we will consider type IIA configurations with non-zero NSNS three-form and RR zero and two-forms. We also take all the sources to be space-time filling and of the same dimension $p = 6$. Since there could be intersecting sources, we will consider a constant dilaton, $e^{\phi} = g_s$, and a constant warp factor.

Assuming additionally that the sources are supersymmetric, one can combine the four- and six-

[1] Actually, for slow roll inflation models, this constraint can be slightly released.

dimensional traces of Einstein equations and the dilaton e.o.m. to get

$$R_4 = \frac{2}{3}(g_s^2|F_0|^2 - |H|^2) \ , \tag{5.1.1}$$

$$R_6 + \frac{1}{2}g_s^2|F_2|^2 + \frac{3}{2}(g_s^2|F_0|^2 - |H|^2) = 0 \, . \tag{5.1.2}$$

The second equation is just a constraint on internal quantities, while the first fixes R_4. From these two equations we recover the minimal requirements of having $F_0 \neq 0$ and $R_6 < 0$. The negative contribution from H is not always easy to balance, since F_0 and H are not independent (they are related via the H e.o.m. and the F_2 Bianchi identity). Adding more fluxes, F_4 and F_6, does not help because they contribute with negative signs. In practice, it turns out that F_0 is often not enough to find a de Sitter vacuum. This is why up to date, all known examples of stable de Sitter vacua require some additional ingredients such as KK monopoles and Wilson lines [71], non-geometric fluxes [76], or α' corrections and probe D6 branes [77].

Here we want to see whether by milden some assumption, it is possible to find de Sitter solutions in classical geometric compactifications. To this purpose, we decide to come back to the assumption of SUSY sources. Since we are interested in non-supersymmetric backgrounds, there is a priori no justification to preserve the supersymmetry of the sources[2]. Therefore, we propose an ansatz for SUSY breaking sources. This will result in a new positive contribution for R_4.

For a supersymmetric source, one can replace the volume form on the brane worldvolume by the pullback of the non-integrable pure spinor [45, 30]

$$\left(i^*[\mathrm{Im}\,\Phi_-] \wedge e^{\mathcal{F}}\right) = \frac{|a|^2}{8}\sqrt{|i^*[g] + \mathcal{F}|}\mathrm{d}^\Sigma x \, , \tag{5.1.3}$$

where i denotes the embedding of the worldvolume into the internal manifold M, g is the internal metric and $\mathcal{F}$ the field strength of the gauge field on the brane worldvolume. To consider non-supersymmetric sources, we propose to modify (5.1.3) to

$$\left(i^*[\mathrm{Im}\,X_-] \wedge e^{\mathcal{F}}\right) = \sqrt{|i^*[g] + \mathcal{F}|}\mathrm{d}^\Sigma x \, , \tag{5.1.4}$$

where X_- is an odd polyform, pulled back from the bulk, that is not pure. X_- consists in a general expansion of an odd form on the basis on $TM \oplus T^*M$ provided by the pure spinors $\Phi_\pm$. For supersymmetric configurations, X_- reduces to Φ_-. The new source term (5.1.4) allows to rewrite (5.1.1) for the four dimensional Ricci scalar as

$$R_4 = \frac{2}{3}\left(\frac{g_s}{2}(T_0 - T) + g_s^2|F_0|^2 - |H|^2\right) \ , \tag{5.1.5}$$

where T is the trace of the energy momentum tensor, and T_0 the supersymmetric part of the trace: for SUSY sources, $T_0 = T$. One can show that $T_0 > 0$, giving a positive contribution to R_4.

Equation (5.1.4) additionally helps to solve the internal Einstein equation with non-supersymmetric fluxes. Indeed, we are able to provide a concrete example of a ten-dimensional de Sitter solution in type IIA supergravity.

In order to further justify this condition we shall show that (with $\mathrm{d}_H = \mathrm{d} - H\wedge$)

$$\begin{aligned} &\mathrm{d}_H(e^{2A-\phi}\,\mathrm{Re}\,X_-) = 0 \, , \\ &\mathrm{d}_H(e^{4A-\phi}\,\mathrm{Im}\,X_-) = c_0 e^{4A} * \lambda(F) \, , \end{aligned} \tag{5.1.6}$$

where c_0 is a constant fixed by the parameters of the solution. These are first order equations which generalize the SUSY conditions (2.4.19) and (2.4.20) on Φ_-. Note that, as for the supersymmetric

[2]This clearly makes it more difficult to check that the brane configuration is stable. We will come back later to the question of the stability of such non-supersymmetric solutions.

case, the second equation in (5.1.6) implies that the RR e.o.m. are automatically satisfied, provided that no NSNS source is present ($\mathrm{d}H = 0$). Indeed, by differentiating either (2.4.20) or (5.1.6), one gets automatically

$$(\mathrm{d} + H\wedge)(e^{4A} * F) = 0 \ . \tag{5.1.7}$$

Note that the presence of c_0, generally not equal to one, tells that, differently from generalized calibrations [45, 30, 46, 11], the source energy density is not minimized. It is the combined bulk-brane energy density that is extremized here.

The idea of solving first order equations to find non-supersymmetric solutions is not new. Nevertheless, a generalisation of the pure spinor equations to study non-supersymmetric backgrounds has been recently proposed in [78]. The idea is to express the violation of the SUSY conditions (2.4.18), (2.4.19), and (2.4.20) as an expansion on the $Spin(6,6)$ basis constructed from the SUSY pure spinors. For instance, for Minkowski compactifications, the modified first order equations are

$$\begin{aligned}
&\mathrm{d}_H(e^{2A-\phi}\Phi_1) = \Upsilon \,, \\
&\mathrm{d}_H(e^{A-\phi}\,\mathrm{Re}\,\Phi_2) = \mathrm{Re}\,\Xi \,, \\
&\mathrm{d}_H(e^{3A-\phi}\,\mathrm{Im}\,\Phi_2) - \frac{|a|^2}{8} e^{3A} * \lambda(F) = \mathrm{Im}\,\Xi \,,
\end{aligned} \tag{5.1.8}$$

where schematically

$$\begin{aligned}
\Upsilon \;&=\; a_0\Phi_2 + \tilde{a}_0\overline{\Phi}_2 + a^1_m\gamma^m\Phi_1 + a^2_m\Phi_1\gamma^m + \tilde{a}^1_m\gamma^m\overline{\Phi}_1 + \tilde{a}^2_m\overline{\Phi}_1\gamma^m \\
&\quad + a_{mn}\gamma^m\Phi_2\gamma^n + \tilde{a}_{mn}\gamma^n\overline{\Phi}_2\gamma^m \,, && (5.1.9) \\
\Xi \;&=\; b_0\,\Phi_1 + \tilde{b}_0\,\overline{\Phi}_1 + b^1_m\gamma^m\Phi_2 + b^2_m\Phi_2\gamma^m + b_{mn}\gamma^m\Phi_1\gamma^n + \tilde{b}_{mn}\gamma^n\overline{\Phi}_1\gamma^m \,. && (5.1.10)
\end{aligned}$$

In the particular case of an $SU(3)$ structure, this decomposition is equivalent to the expansion in the $SU(3)$ torsion classes (4.4.52). This idea has been used to look for non-supersymmetric solutions on Minkowski and Anti de Sitter [78, 72, 79, 80, 75]. However this approach assumes that the $4d$ supersymmetry is not explicitly broken and that the breaking only take place in the internal manifold. For this reason it does not apply directly to de Sitter compactifications.

Our explicit de Sitter solution is found on the solvmanifold $\mathfrak{g}^{p,-p,\pm1}_{5.17} \times S^1$ of algebra $(q_1(p25 + 35), q_2(p15 + 45), q_2(p45 - 15), q_1(p35 - 25), 0, 0)$. As explained in section 4.2, for $p = 0$, this algebra reduces to s 2.5, while for $p \neq 0$, the manifold admits a supersymmetric solution provided a certain combination of moduli, which we call λ, is equal to one. For generic λ, the pure spinor equations are not satisfied and supersymmetry is broken. This set-up will serve as an ansatz to find a de Sitter solution. It is certainly of great practical importance to have a SUSY limit in which our construction can be tested.

In the rest of this chapter we present in more detail the treatment of supersymmetry breaking branes and the explicit form of the de Sitter solution. We also compute and study the four-dimensional effective potential. In particular, we will discuss how the non-supersymmetric sources contribute to new terms in the potential. We will also provide an analysis of stability of the solution in the volume and dilaton moduli. The stability in the other moduli remains to be studied. Similarly, a complete discussion whether the proposal (5.1.4) and (5.1.6) can provide stable sources is postponed to further work.

5.2 Ten-dimensional analysis

5.2.1 Action and equations of motion

We consider type IIA supergravity and follow the conventions[3] of section 2.1. In order to derive the ten-dimensional equations of motion, we shall need the source terms of the action, that we have not specified so far. To this end let us consider the DBI action of only one Dp-brane in string frame

$$S_s = -T_p \int \mathrm{d}^{p+1}x \; e^{-\phi}\sqrt{|i^*[g_{10}] + \mathcal{F}|} \,, \qquad T_p^2 = \frac{\pi}{\kappa^2}(4\pi^2\alpha')^{3-p} \,.$$

Here T_p is the tension of the brane; for an O-plane, one has to replace T_p by $-2^{p-5}T_p$. $\mathcal{F}$ is the field strength of the gauge field on the brane worldvolume. The open string excitations will not be important for our solution, and we shall discard the $\mathcal{F}$ contribution from now on. However, the fact that the brane wraps a non-trivial cycle is going to be important, and to derive the equations of motion, a priori, we should take a full variation of the DBI action with respect to the bulk metric.

For supersymmetric (calibrated) sources it exists a convenient way of avoiding this. In this case, one can think of an expansion of the DBI action around the supersymmetric configuration and, to leading order, replace the DBI action by a pullback of the calibration form. This is given in terms of the non-closed pure spinor (Φ_- in type IIA) as given in (5.1.3). As shown in [11], this allows to prove that, for Minkowski compactifications, Einstein equations follow from the first order pure spinor equations and Bianchi identities. A similar treatment of space-time filling sources is also possible for non-supersymmetric Minkowski and AdS_4 configurations [78]. It is worth stressing that, even in these cases, the sources continue being (generalized) calibrated and are not responsible for the supersymmetry breaking. However convenient, as we shall see, these kinds of source are not going to be helpful in our search for a dS vacuum.

At this point we shall make an important simplifying assumption, which will be justified a posteriori and, for now, can be thought of as a generalization of the calibrated sources to non-supersymmetric backgrounds. The backgrounds in question are of somewhat restricted type: we shall consider the case of an internal space with $SU(3)$ structure. We shall assume that, in analogy with the supersymmetric case, the DBI action can be replaced to leading order by the pullback of a (poly)form X in the bulk. The bulk does have invariant forms and hence pure spinors can be constructed, but X cannot be pure, otherwise the source will be supersymmetric. The form X is expandable in the Hodge diamond defined by the pure spinors. This amounts to consider forms that are equivalent not to simply the invariant spinor η_+ (defining the SU(3) structure) but to a full spinorial basis, η_+, η_-, $\gamma^{\bar{i}}\eta_+$ and $\gamma^i\eta_-$, where $i, \bar{i} = 1, \ldots 3$ are the internal holomorphic and antiholomorphic indices[4]. To be concrete we shall consider a generic odd form

$$\begin{aligned}
X &= \sqrt{|g_4|}\; \mathrm{d}^4x \wedge X_- = \sqrt{|g_4|}\; \mathrm{d}^4x \wedge (\mathrm{Re}\, X_- + i\, \mathrm{Im}\, X_-)\,,\\
X_- &= \mathrm{Re}\, X_- + i\, \mathrm{Im}\, X_- = \frac{8}{||\Phi_-||}\Big(\alpha_0\Phi_- + \tilde{\alpha}_0\overline{\Phi}_- + \alpha_{mn}\gamma^m\Phi_-\gamma^n + \tilde{\alpha}_{mn}\gamma^m\overline{\Phi}_-\gamma^n \\
&\qquad + \alpha^L_m\gamma^m\Phi_+ + \tilde{\alpha}^L_m\gamma^m\overline{\Phi}_+ + \alpha^R_n\Phi_+\gamma^n + \tilde{\alpha}^R_n\overline{\Phi}_+\gamma^n\Big)\,, \qquad (5.2.2)
\end{aligned}$$

[3] These conventions are consistent with the SUSY conditions written before. Note we have a factor of 2 difference in the normalisation of the RR kinetic terms with respect to [71], which will result in a difference in the RR quantization conditions. For a k-flux α through a k-cycle Σ (with embedding i into the bulk manifold M), we have

$$\frac{1}{(2\pi\sqrt{\alpha'})^{k-1}}\frac{1}{\mathrm{vol}_M}\int_\Sigma i^*\alpha = \frac{1}{(2\pi\sqrt{\alpha'})^{k-1}}\frac{1}{\mathrm{vol}_M}\int_M \langle\delta(\Sigma\hookrightarrow M), \alpha\rangle = n\,, \qquad (5.2.1)$$

where n is an integer.

[4] The covariant derivative on the invariant spinor contains the same information as the intrinsic torsions. For the explicit dictionary for $SU(3)$ structure see [28]. In the supersymmetric backgrounds the (H-twisted) derivative on the spinor cancels against the RR contribution [4], and the entire content of that cancellation is captured by first order equation on the pure spinors (2.4.18), (2.4.19), and (2.4.20). For the non-supersymmetric backgrounds, the unbalance between the NSNS and RR contributions results in the presence of terms that need to be expanded in the full basis as in (5.1.8).

where $\Phi_\pm$ are given in (2.4.11) (we take $|a|=1$, $\theta_\pm=0$) and the γ's act on even and odd forms via contractions and wedges

$$\gamma^m \Phi_\pm = (g^{mn}\imath_n + \mathrm{d}x^m\wedge)\,\Phi_\pm\,, \qquad \text{and} \qquad \Phi_\pm\gamma^m = \mp(g^{mn}\imath_n - \mathrm{d}x^m\wedge)\,\Phi_\pm\,. \tag{5.2.3}$$

The action for a single source term becomes

$$\begin{aligned} S_s &= -T_p \int_\Sigma \mathrm{d}^{p+1}x\; e^{-\phi}\sqrt{|i^*[g_{10}]|} \\ &= -T_p \int_\Sigma \; e^{-\phi} i^*[\mathrm{Im}\, X] \\ &= -T_p \int_{M_{10}} \; e^{-\phi}\langle j_p, \mathrm{Im}\, X\rangle \\ &= T_p \int_{M_{10}} \mathrm{d}^{10}x\sqrt{|g_{10}|}\; e^{-\phi}\hat{*}\langle j_p, \mathrm{Im}\, X\rangle\ , \end{aligned} \tag{5.2.4}$$

where $i:\Sigma\hookrightarrow M_{10}$ is the embedding of the cycle in the bulk and $j_p=\delta(\Sigma\hookrightarrow M_{10})$ is the dimensionless Poincaré dual[5] of the cycle Σ. The change of sign between the last two lines is due to the Lorentzian signature which gives a minus when taking the Hodge star. We recall we denote by $\hat{*}$ the ten-dimensional Hodge star, and $*$ its six-dimensional counterpart. For the sum of all sources we then take the action

$$S_s = T_p \int_{M_{10}} \mathrm{d}^{10}x\sqrt{|g_{10}|}\; e^{-\phi}\hat{*}\langle j, \mathrm{Im}\, X\rangle\ , \qquad\qquad j=\sum_{Dp} j_p - \sum_{Op} 2^{p-5} j_p\,. \tag{5.2.5}$$

As already mentioned in the introduction, we will consider solutions where the only non-trivial fluxes are H, F_0 and F_2 on the internal manifold, and the RR magnetic sources are $D6$'s and $O6$'s. The sources will be smeared, so we take $\delta\to 1$ and the warp factor $e^{2A}=1$. Out of section 2.1, we get that the relevant part of the action[6], in string frame, is then

$$S=\frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\sqrt{|g_{10}|}\,[e^{-2\phi}(R_{10}+4|\nabla\phi|^2-\frac{1}{2}|H|^2)-\frac{1}{2}(|F_0|^2+|F_2|^2)+2\kappa^2 T_p\, e^{-\phi}\hat{*}\langle j, \mathrm{Im}\, X\rangle]\ . \tag{5.2.6}$$

Given the assumptions made on the fluxes, their equations of motion (e.o.m.) and Bianchi identities worked out in sections 2.1 and 2.2.1 reduce to

$$\begin{aligned} &\mathrm{d}H = 0\,, \\ &\mathrm{d}F_0 = 0\ , \\ &\mathrm{d}F_2 - H\wedge F_0 = 2\kappa^2 T_p\, j\ , \\ &H\wedge F_2 = 0\ , \\ \\ &\mathrm{d}(e^{-2\phi} * H) = -F_0\wedge *F_2 - e^{-\phi}\, 4\kappa^2 T_p\, j\wedge \mathrm{Im}\, X_1\,, \\ &\mathrm{d}(*F_2) = 0\,, \end{aligned}$$

[5] The fact we put j_p on the left is related to our conventions. See appendix B.3.

[6] By relevant we mean the parts of the bulk and source actions that give non-trivial contributions to the Einstein and dilaton equations of motion and to the derivation of the four-dimensional effective potential of section 5.4. We do not write down the Chern-Simons terms of the bulk action and the Wess-Zumino part of the source action. Indeed they do not have any metric nor dilaton dependence and, since we do not allow for non-zero values of RR gauge potentials in the background, they will not contribute to the vacuum value of the four-dimensional potential either. However, both terms contribute the flux e.o.m. and Bianchi identities (in particular, see [81, 82, 10] for a discussion of the Chern-Simons terms in the presence of non-trivial background fluxes).

where $\mathrm{Im}\, X_1$ is the one-form part[7] of $\mathrm{Im}\, X_-$ in (5.2.2). Similarly, the ten-dimensional Einstein and dilaton equations of section 2.1 (in string frame) become

$$\begin{aligned} R_{MN} - \frac{g_{MN}}{2} R_{10} = -4\nabla_M\phi\nabla_N\phi + \frac{1}{4} H_{MPQ} H_N{}^{PQ} + \frac{e^{2\phi}}{2} F_{2\ MP} F_{2\ N}{}^P \\ -\frac{g_{MN}}{2}\left(-4|\nabla\phi|^2 + \frac{1}{2}|H|^2 + \frac{e^{2\phi}}{2}(|F_0|^2 + |F_2|^2)\right) + e^{\phi}\ \frac{1}{2} T_{MN}\ , \end{aligned} \quad (5.2.7)$$

$$8(\nabla^2\phi - |\nabla\phi|^2) + 2R_{10} - |H|^2 = -e^{\phi}\frac{T_0}{p+1}\,. \quad (5.2.8)$$

We recall that T_{MN} and T_0 are the source energy momentum tensor and its partial trace, respectively. They are defined in footnote 2 of chapter 2. Out of the explicit expression for the source action, we compute[8]

$$\begin{aligned} T_{MN} &= 2\kappa^2 T_p\, \hat{*}\langle j, g_{P(M} dx^P \otimes \iota_{N)}\, \mathrm{Im}\, X\ - \delta^m_{(M} g_{N)n} C_m{}^n\rangle\,, && (5.2.9)\\ T_0 &= 2\kappa^2 T_p\, \hat{*}\langle j, dx^N \otimes \iota_N\, \mathrm{Im}\, X\rangle = (p+1)\ 2\kappa^2 T_p \hat{*}\langle j, \mathrm{Im}\, X\rangle\,, && (5.2.10)\\ T &= g^{MN} T_{MN} = T_0\ - 2\kappa^2 T_p\, \hat{*}\langle j, C_m{}^m\rangle\ . && (5.2.11) \end{aligned}$$

m, n are real internal indices, $C_m{}^n = \sqrt{|g_4|}\ \mathrm{d}^4x \wedge c_m{}^n$ and

$$\begin{aligned} c_m{}^n &= \frac{8}{||\Phi_-||}\,\mathrm{Im}\left(\alpha^L_m \gamma^n \Phi_+ + \tilde{\alpha}^L_m \gamma^n \overline{\Phi}_+ + \alpha^R_m \Phi_+ \gamma^n + \tilde{\alpha}^R_m \overline{\Phi}_+ \gamma^n \right.\\ &\left. +\alpha_{pm}\gamma^p \Phi_- \gamma^n + \alpha_{mp}\gamma^n \Phi_- \gamma^p + \tilde{\alpha}_{pm}\gamma^p \overline{\Phi}_- \gamma^n + \tilde{\alpha}_{mp}\gamma^n \overline{\Phi}_- \gamma^p\right)\ . && (5.2.12) \end{aligned}$$

For supersymmetric branes $\mathrm{Im}\, X_- = 8\ \mathrm{Im}\Phi_-$, $c_m{}^n = 0$, T_0 reduces to the full trace of the source energy-momentum tensor, $T = T_0$ and one recovers the formulae in [11].

We can now split (5.2.7) into its four and six-dimensional components. Since for maximally symmetric spaces, $R_{\mu\nu} = \Lambda g_{\mu\nu} = (R_4/4) g_{\mu\nu}$, for constant dilaton, $e^\phi = g_s$, the four-dimensional Einstein equation has only one component and reduces to

$$R_4 = -2R_6 + |H|^2 + g_s^2(|F_0|^2 + |F_2|^2) - 2g_s\tilde{T}_0 = 4\Lambda\ . \quad (5.2.13)$$

Not to clutter equations, from now on, we set $\tilde{T}_0 = T_0/(p+1)$. This equation defines the cosmological constant, Λ. Using the dilaton equation (5.2.8), the source contribution can be eliminated and we obtain

$$\begin{aligned} R_4 &= \frac{2}{3}[-R_6 - \frac{g_s^2}{2}|F_2|^2 + \frac{1}{2}(|H|^2 - g_s^2|F_0|^2)]\,, && (5.2.14)\\ R_{10} &= \frac{1}{3}[R_6 + |H|^2 - g_s^2(|F_0|^2 + |F_2|^2)]\,. && (5.2.15) \end{aligned}$$

We are left with the internal Einstein equation,

$$R_{mn} - \frac{1}{4} H_{mpq} H_n{}^{pq} - \frac{g_s^2}{2} F_{2\ mp} F_{2\ n}{}^p - \frac{g_{mn}}{6}[R_6 - \frac{1}{2}|H|^2 - \frac{5}{2} g_s^2(|F_0|^2 + |F_2|^2)] = \frac{g_s}{2} T_{mn}\ , \quad (5.2.16)$$

and the dilaton equation

$$g_s\tilde{T}_0 = \frac{1}{3}[-2R_6 + |H|^2 + 2g_s^2(|F_0|^2 + |F_2|^2)]\,. \quad (5.2.17)$$

[7] We refer to [11] for a discussion of the last term in the H equation of motion.

[8] To derive (5.2.9), we considered the fact that each γ_m matrix in the bispinors $\Phi_\pm$ carries one vielbein. To derive $C_m{}^n$ the metric dependence of the full Hodge decomposition (5.2.2) must be taken into account. For supersymmetric cases, the operator $g_{P(M} dx^P \otimes \iota_{N)}$ in T_{MN} is the projector on the cycle wrapped by the source [13].

Provided the flux equations of motion and Bianchi identities are satisfied, solving the Einstein and dilaton equations becomes equivalent to finding the correct energy-momentum tensor for the sources. We shall now consider an explicit example and see how the non-supersymmetric modifications to the energy momentum tensor help in looking for de Sitter solutions. In the process we shall establish some properties of the calibrating form $\mathrm{Im}\, X_-$.

5.2.2 Solvable de Sitter

Ansatz of solution

Our starting point is the solution described in section 4.4.3, based on the algebra

$$(q_1(p25+35), q_2(p15+45), q_2(p45-15), q_1(p35-25), 0, 0)\ . \tag{5.2.18}$$

Among the different O6 projections compatible with the algebra for $p=0$, only those along 146 or 236 are still compatible with the full algebra with $p \neq 0$. In section 4.4.3 we showed that, acting with a twist transformation on the supersymmetric solution with $p=0$ and the right O6 planes, one finds a family of backgrounds characterized by the $SU(3)$ structure

$$\Omega = \sqrt{t_1 t_2 t_3}(e^1 + i\lambda \frac{\tau_3}{\tau_4} e^2) \wedge (\tau_3\ e^3 + i\tau_4\ e^4) \wedge (e^5 - i\tau_6\ e^6)\,, \tag{5.2.19}$$

$$J = t_1 \lambda \frac{\tau_3}{\tau_4} e^1 \wedge e^2 + t_2 \tau_3 \tau_4 e^3 \wedge e^4 - t_3 \tau_6 e^5 \wedge e^6\,, \tag{5.2.20}$$

which satisfy the supersymmetry equations (2.4.18) and (2.4.19) only when the parameter $\lambda = \frac{t_2 \tau_4^2}{t_1}$ is equal to one. One motivation to consider what happens when supersymmetry is violated comes from the form of the Ricci scalar for this class of backgrounds[9]

$$R_6 = -\frac{1}{t_1 t_2 t_3 \tau_3^2}\left[(A-B)^2 + p^2\left(\frac{(\lambda-1)^2}{2\lambda}(A^2+B^2) + (A+B)^2\right)\right]\ , \tag{5.2.24}$$

where we introduced the following quantities

$$A = q_1 t_1 \qquad B = q_2 t_2 \tau_3^2\ . \tag{5.2.25}$$

Indeed, R_6 gets more negative when the SUSY breaking parameters p and $|\lambda - 1|$ leave their SUSY value 0. Therefore, the value R_4 as given in (5.2.14) is lifted by SUSY breaking and this is a priori promising for a de Sitter vacuum.

[9] The Ricci tensor of a group manifold is easily computed in frame indices (where the metric is the unit one) in terms of the group structure constants

$$R_{ad} = \frac{1}{2}\left(\frac{1}{2} f_a{}^{bc} f_{dbc} - f^c{}_{db} f_{ca}{}^b - f^b{}_{ac} f^c{}_{db}\right)\ . \tag{5.2.21}$$

In our case, with the appropriate rescaling of the one-forms e^a in (4.4.43) and of the structure constants, we find that the only non-zero components of the Ricci tensor are

$$\begin{aligned}
R_{11} = -R_{22} &= \frac{1}{2t_1 t_2 t_3 \tau_3^2}\left[A^2 - B^2 + \frac{p^2}{\lambda}(A^2 - \lambda^2 B^2)\right]\ , \\
R_{33} = -R_{44} &= \frac{1}{2t_1 t_2 t_3 \tau_3^2}\left[B^2 - A^2 + \frac{p^2}{\lambda}(B^2 - \lambda^2 A^2)\right]\ , \\
R_{55} &= -\frac{1}{t_1 t_2 t_3 \tau_3^2}\left[(A-B)^2 + p^2\left(\frac{1+\lambda^2}{2\lambda}(A^2+B^2) + 2AB\right)\right]\ , && (5.2.22) \\
R_{14} = R_{23} &= \frac{1}{2t_1 t_2 t_3 \tau_3^2}\frac{p}{\sqrt{\lambda}}(\lambda-1)(A^2 - B^2)\,. && (5.2.23)
\end{aligned}$$

Notice that the curvature only receives contributions from R_{55}.

The rest of this section is devoted to the search of de Sitter solutions on the class of backgrounds discussed above. We will take the same $SU(3)$ structure as in (5.2.19) and metric

$$g = \mathrm{diag}\left(t_1, \lambda t_2\tau_3^2, t_2\tau_3^2, \lambda t_1, t_3, t_3\tau_6^2\right) \tag{5.2.26}$$

in the basis of e^m given in (4.4.43). Dilaton and warp factor are still constant: $e^\phi = g_s$ and $e^{2A} = 1$. For the fluxes, beside the RR two-form, we will allow for non-trivial RR zero-form and NSNS three-form

$$\begin{aligned}
H &= h\,(t_1\sqrt{t_3\lambda}\, e^1\wedge e^4\wedge e^5 + t_2\tau_3^2\sqrt{t_3\lambda}\; e^2\wedge e^3\wedge e^5)\,,\\
g_sF_2 &= \gamma\sqrt{\frac{\lambda}{t_3}}\left[(A-B)(e^3\wedge e^4 - e^1\wedge e^2) + \frac{p}{\lambda}(A+B)(\lambda^2\; e^2\wedge e^4 + e^1\wedge e^3)\right]\,,\\
g_sF_0 &= \frac{h}{\gamma}\,.
\end{aligned} \tag{5.2.27}$$

The NSNS flux has component along the covolumes, $v^1 = t_1\sqrt{t_3\lambda}\, e^1\wedge e^4\wedge e^5$ and $v^2 = t_2\tau_3^2\sqrt{t_3\lambda}\; e^2\wedge e^3\wedge e^5$ of the calibrated sources in the supersymmetric case[10]. We have introduced here another parameter $\gamma > 0$ which is given by the ratio of NSNS and RR zero-form fluxes.

The SUSY solutions of section 4.4.3 are obtained setting

$$\lambda = 1 \text{ or } p = 0,\;\; \gamma = 1\;,F_0 = h = 0\;. \tag{5.2.28}$$

The solution

We will first consider the four-dimensional Einstein equation (5.2.14). Using the ansatz for the fluxes we obtain

$$\begin{aligned}
g_s^2|F_2|^2 &= \frac{2\gamma^2}{t_1t_2t_3\tau_3^2}\left[(A-B)^2 + p^2(A+B)^2\left(\frac{(\lambda-1)^2}{2\lambda}+1\right)\right]\;,\\
|H|^2 &= 2h^2\;.
\end{aligned} \tag{5.2.29}$$

Notice that

$$g_s^2|F_2|^2 = 2\gamma^2\left[-R_6 + p^2\frac{(\lambda-1)^2}{\lambda}\,\frac{q_1q_2}{t_3}\right]\;. \tag{5.2.30}$$

This allows to write the four dimensional Ricci scalar as

$$R_4 = \frac{2}{3}\left[(1-2\gamma^2)(-R_6 - \frac{1}{2}g_s^2|F_0|^2) + \gamma^2\left(-R_6 - \frac{q_1q_2}{t_3}\;p^2\frac{(\lambda-1)^2}{\lambda}\right)\right]\;. \tag{5.2.31}$$

Since the second bracket is positive

$$-R_6 - \frac{q_1q_2}{t_3}\;p^2\frac{(\lambda-1)^2}{\lambda} = \frac{q_1q_2}{t_3}\frac{1}{AB}\left[(A-B)^2 + p^2\left(\frac{(\lambda-1)^2}{2\lambda}(A-B)^2 + (A+B)^2\right)\right] > 0\;, \tag{5.2.32}$$

we see that de Sitter solutions are possible, for instance, for $\gamma^2 \le \frac{1}{2}$ and small F_0. Note also that R_4 clearly vanishes in the supersymmetric solution where $\lambda = 1$, $\gamma = 1$ and $F_0 = 0$.

[10]In order not to clutter the notations we did not divide v^i by $\sqrt{2}$ (and recalibrate the cycles accordingly) with an unfortunate consequence that H in the normalisation discussed in footnote 3 comes out as even-quantized, and γ is rational up to multiplication by $\sqrt{2}$.

To solve the dilaton and internal Einstein equations it is more convenient to go to frame indices and take a unit metric. As already discussed in footnote 9, this choice makes the computation of the Ricci tensor very simple. To simplify notations we introduce the constant

$$C = -\frac{1}{6}\left(R_6 - \frac{1}{2}|H|^2 - \frac{5}{2}g_s^2(|F_0|^2 + |F_2|^2)\right) \ . \tag{5.2.33}$$

Then the dilaton equation becomes

$$g_s\tilde{T}_0 = 4C - \frac{h^2}{\gamma^2} - \frac{2\gamma^2}{t_1t_2t_3\tau_3^2}\left[(A-B)^2 + p^2(A+B)^2\left(\frac{(\lambda-1)^2}{2\lambda}+1\right)\right] \ . \tag{5.2.34}$$

For the internal Einstein equations, only some components are non-trivial

$$\begin{aligned}
g_sT_{14} &= \frac{1}{t_1t_2t_3\tau_3^2}\frac{p}{\sqrt{\lambda}}(A^2-B^2)(\lambda-1)(1-\gamma^2)\,,\\
g_sT_{23} &= \frac{1}{t_1t_2t_3\tau_3^2}\frac{p}{\sqrt{\lambda}}(A^2-B^2)(\lambda-1)(1-\gamma^2)\,,\\
g_sT_{11} &= \frac{1}{t_1t_2t_3\tau_3^2}\left[A^2-B^2+\frac{p^2}{\lambda}(A^2-B^2\lambda^2)-\gamma^2((A-B)^2+\frac{p^2}{\lambda}(A+B)^2)\right]-h^2+2C\,,\\
g_sT_{22} &= \frac{1}{t_1t_2t_3\tau_3^2}\left[B^2-A^2+\frac{p^2}{\lambda}(B^2\lambda^2-A^2)-\gamma^2((A-B)^2+p^2\lambda(A+B)^2)\right]-h^2+2C\,,\\
g_sT_{33} &= \frac{1}{t_1t_2t_3\tau_3^2}\left[B^2-A^2+\frac{p^2}{\lambda}(B^2-A^2\lambda^2)-\gamma^2((A-B)^2+\frac{p^2}{\lambda}(A+B)^2)\right]-h^2+2C\,,\\
g_sT_{44} &= \frac{1}{t_1t_2t_3\tau_3^2}\left[A^2-B^2+\frac{p^2}{\lambda}(A^2\lambda^2-B^2)-\gamma^2\left((A-B)^2+p^2\lambda(A+B)^2\right)\right]-h^2+2C\,,\\
g_sT_{55} &= -\frac{2}{t_1t_2t_3\tau_3^2}\left[(A-B)^2+p^2\left(\frac{(\lambda^2+1)}{2\lambda}(A^2+B^2)+2AB\right)\right]-2h^2+2C\,,\\
g_sT_{66} &= 2C\,.
\end{aligned} \tag{5.2.35}$$

The remaining components set to zero the corresponding source term $T_{ab}=0$.

To solve these equations we need the explicit expressions for the source energy momentum tensor, (5.2.9). In six-dimensional frame indices we have

$$\begin{aligned}
T_{ab} &= 2\kappa^2T_p\hat{*}\langle j, \delta_{c(a}e^c\otimes\iota_{b)}\,\mathrm{Im}\,X \ -\delta^c_{(a}\delta_{b)d}C^d_c\rangle\\
&= 2\kappa^2T_p\hat{*}\left(\sqrt{|g_4|}\ \mathrm{d}^4x\wedge\langle j, \delta_{c(a}e^c\otimes\iota_{b)}\,\mathrm{Im}\,X_- \ -\delta^c_{(a}\delta_{b)d}c^d_c\rangle\right)\\
&= 2\kappa^2T_p\frac{1}{\sqrt{|g_6|}}\ \left[j\wedge\left(\delta_{c(a}e^c\otimes\iota_{b)}\,\mathrm{Im}\,X_3 \ -\delta^c_{(a}\delta_{b)d}c^d_c|_3\ \right)\right]_{1\ldots6}\\
&= \frac{1}{\sqrt{|g_6|}}\ \left[(\mathrm{d}F_2-HF_0)\wedge\left(\delta_{c(a}e^c\otimes\iota_{b)}\,\mathrm{Im}\,X_3 \ -\delta^c_{(a}\delta_{b)d}c^d_c|_3\ \right)\right]_{1\ldots6}\ .
\end{aligned} \tag{5.2.36}$$

Since, in our case, the source j is a three-form,

$$2\kappa^2T_p\ j = \mathrm{d}F_2 - HF_0\,, \tag{5.2.37}$$

only the three-form parts $\mathrm{Im}\,X_3$ and $c^d_c|_3$ of $\mathrm{Im}\,X_-$ and c^d_c contribute to the equations. In the same way, we obtain

$$g_s\tilde{T}_0 = g_s\ 2\kappa^2T_p\,\hat{*}\langle j,\mathrm{Im}\,X\rangle = \frac{1}{\sqrt{|g_6|}}\ \left[g_s\,(\mathrm{d}F_2-HF_0)\wedge\mathrm{Im}\,X_3\right]_{1\ldots6}\ . \tag{5.2.38}$$

Combining (5.2.2) and the explicit expression for $SU(3)$ pure spinors, it is easy to see that $\mathrm{Im}\, X_-$ decomposes into a one-form, a three-form and a five-form piece

$$\mathrm{Im}\, X_- = \mathrm{Im}\, X_1 + \mathrm{Im}\, X_3 + \mathrm{Im}\, X_5\,, \tag{5.2.39}$$

where[11]

$$\begin{aligned}
\mathrm{Im}\, X_1 &= (a_k^{i\,L} + a_k^{i\,R})\mathrm{d}x^k - (a_k^{r\,L} - a_k^{r\,R})g^{kj}\iota_j J + (g^{km}g^{jl}\iota_m\iota_l)[-a^r_{kj}\,\mathrm{Re}\,\Omega + a^i_{kj}\,\mathrm{Im}\,\Omega]\,,\\
\mathrm{Im}\, X_3 &= -(a_k^{r\,L} + a_k^{r\,R})\mathrm{d}x^k \wedge J - (a_k^{i\,L} - a_k^{i\,R})\, g^{kj}\iota_j J\wedge J\\
&- [a^r_0 - a^r_{kj}\,(g^{kj} - (g^{kl}\mathrm{d}x^j + g^{jl}\mathrm{d}x^k)\iota_l)]\,\mathrm{Re}\,\Omega\\
&+ [a^i_0 - a^i_{kj}(g^{kj} - (g^{kl}\mathrm{d}x^j + g^{jl}\mathrm{d}x^k)\iota_l)]\,\mathrm{Im}\,\Omega\,,\\
\mathrm{Im}\, X_5 &= -\frac{1}{2}[(a_k^{i\,L} + a_k^{i\,R}))\mathrm{d}x^k - (a_k^{r\,L} - a_k^{r\,R}))g^{kj}\iota_j J]\wedge J^2\\
&- \mathrm{d}x^k\wedge\mathrm{d}x^j\wedge[-a^r_{kj}\,\mathrm{Re}\,\Omega + a^i_{kj}\,\mathrm{Im}\,\Omega]\,.
\end{aligned} \tag{5.2.40}$$

The superscripts r and i indicate real and imaginary parts:

$$\begin{aligned}
a^r_0 &= \mathrm{Re}(\alpha_0 - \widetilde{\alpha}_0)\,, &\qquad a^r_{jk} &= \mathrm{Re}(\alpha_{jk} - \widetilde{\alpha}_{jk})\,,\\
a^i_0 &= \mathrm{Im}(\alpha_0 + \widetilde{\alpha}_0)\,, &\qquad a^i_{jk} &= \mathrm{Im}(\alpha_{jk} + \widetilde{\alpha}_{jk})\,.
\end{aligned} \tag{5.2.41}$$

and

$$\begin{aligned}
a_k^{r\,L} &= \mathrm{Re}(\alpha_k^L - \widetilde{\alpha}_k^L)\,, &\qquad a_k^{r\,R} &= \mathrm{Re}(\alpha_k^R - \widetilde{\alpha}_k^R)\,,\\
a_k^{i\,L} &= \mathrm{Im}(\alpha_k^L + \widetilde{\alpha}_k^L)\,, &\qquad a_k^{i\,R} &= \mathrm{Im}(\alpha_k^R + \widetilde{\alpha}_k^R)\,.
\end{aligned} \tag{5.2.42}$$

As already discussed, only the three-form parts of $\mathrm{Im}\, X_-$ and c_c^d contribute to the equations. Then, for simplicity, we choose to set to zero $\mathrm{Im}\, X_1$ and $\mathrm{Im}\, X_5$. This amounts to setting

$$a_k^{r\,L} = a_k^{i\,L} = a_k^{r\,R} = a_k^{i\,R} = 0\,, \tag{5.2.43}$$

and choosing a^r_{jk} and a^i_{jk} symmetric. Then, in frame indices, $\mathrm{Im}\, X_3$ becomes

$$\begin{aligned}
\mathrm{Im}\, X_3 &= [a^i_0 - Tr(a^i_{bc}) + a^i_{bc}(\delta^{bd}e^c + \delta^{cd}e^b)\iota_d]\,\mathrm{Im}\,\Omega\\
&- [a^r_0 - Tr(a^r_{bc}) + a^r_{bc}(\delta^{bd}e^c + \delta^{cd}e^b)\iota_d]\,\mathrm{Re}\,\Omega\,.
\end{aligned} \tag{5.2.44}$$

Similarly, we find that the three-form part of c_a^b is given by

$$\begin{aligned}
c_a^b|_3 &= 2a^i_{ac}[-\delta^{bc} + (\delta^{cd}e^b + \delta^{bd}e^c)\iota_d]\,\mathrm{Im}\,\Omega\\
&- 2a^r_{ac}[-\delta^{bc} + (\delta^{cd}e^b + \delta^{bd}e^c)\iota_d]\,\mathrm{Re}\,\Omega\,.
\end{aligned} \tag{5.2.45}$$

The coefficients in $\mathrm{Im}\, X_3$ are free parameters which should be fixed by solving the dilaton and internal Einstein equations.

The equations $T_{mn} = 0$ are satisfied by choosing[12]

$$\begin{aligned}
a^i_0 &= 0 &\qquad & a = 1,\ldots,6\,,\\
a^i_{bc} &= 0 &\qquad & b,c = 1,\ldots,6\,,\\
a^r_{bc} &= 0 &\qquad & (bc)\notin\{(bb),\ (14),\ (23)\}\,.
\end{aligned} \tag{5.2.46}$$

[11]We have not imposed (5.2.4) yet, and shall return to it later.

[12]The parameters a^i_{12}, a^i_{13}, a^i_{24}, a^i_{34}, a^i_{56} are not fixed by any equation. For simplicity, we decide to put them to zero.

The Einstein and dilaton equations, (5.2.35) and (5.2.34) fix the other parameters

$$
\begin{aligned}
a^r_0 &= -g_s\,\frac{\tilde{T}_0+T_{55}+T_{66}-x_0}{2(c_1+c_2)}\,,\\
a^r_{14} &= g_s\,\frac{T_{14}}{2(c_2-c_1)}\,,\\
a^r_{23} &= g_s\,\frac{T_{23}}{2(c_1-c_2)}\,,\\
a^r_{11} &= g_s\,\frac{1}{2(c_2-c_1)}\left[T_{11}-\frac{c_2\tilde{T}_0}{c_1+c_2}+\frac{x_0c_1c_2}{(c_1^2-c_2^2)}\right],\\
a^r_{22} &= g_s\,\frac{1}{2(c_1-c_2)}\left[T_{22}-\frac{c_1\tilde{T}_0}{c_1+c_2}+\frac{x_0c_1c_2}{(c_2^2-c_1^2)}\right],\\
a^r_{33} &= g_s\,\frac{1}{2(c_1-c_2)}\left[T_{33}-\frac{c_1\tilde{T}_0}{c_1+c_2}+\frac{x_0c_1c_2}{(c_2^2-c_1^2)}\right],\\
a^r_{44} &= g_s\,\frac{1}{2(c_2-c_1)}\left[T_{44}-\frac{c_2\tilde{T}_0}{c_1+c_2}+\frac{x_0c_1c_2}{(c_1^2-c_2^2)}\right],\\
a^r_{55} &= -g_s\,\frac{T_{55}}{2(c_1+c_2)}\,,\\
a^r_{66} &= g_s\,\frac{T_{66}-\tilde{T}_0}{2(c_1+c_2)}\,,
\end{aligned}
\tag{5.2.47}
$$

where $x_0 = 2\tilde{T}_0-(T_{11}+T_{22}+T_{33}+T_{44})$ and T_{ab} are given by (5.2.35). The coefficients c_1 and c_2 appear in the source term of the Bianchi identity for F_2 (see (3.3.5))

$$
g_s(\mathrm{d}F_2-HF_0)=c_1\;v^1+c_2\;v^2\,, \tag{5.2.48}
$$

where v^1 and v^2 are covolumes of sources in the directions (146) and (236) and

$$
\begin{aligned}
c_1 &= -\frac{h^2}{\gamma}+\frac{q_1q_2}{At_3}\gamma\left[2(A-B)-p^2\frac{\lambda^2+1}{\lambda}(A+B)\right],\\
c_2 &= -\frac{h^2}{\gamma}+\frac{q_1q_2}{Bt_3}\gamma\left[2(B-A)-p^2\frac{\lambda^2+1}{\lambda}(A+B)\right].
\end{aligned}
\tag{5.2.49}
$$

Whether the sources are D6 branes or O6 planes depends again on the parameters in the solution, but the overall tension is always negative and so is c_1+c_2.

In order to have a solution, we should also solve the equations of motion and the remaining Bianchi identities for the fluxes. However, these are automatically satisfied by our ansatz for the fluxes, provided $j\wedge\mathrm{Im}\,X_1=0$. This condition is satisfied by our choice of the parameters a. Thus we have constructed a de Sitter solution.

5.3 A first step towards first order equations

In this section, we will try to provide further justification for our choice of calibrating polyform X_-. In supersymmetric compactifications, the imaginary part of the non-closed pure spinor, Φ_- in type IIA, on one side, defines the calibration for the sources and, on the other, gives the bulk RR fields in the supersymmetry equation (2.4.20). We will show that, for our de Sitter solution, the polyform X_- satisfies the same equations Φ_- satisfies in the supersymmetric case

$$
\begin{aligned}
(\mathrm{d}-H)\,\mathrm{Re}\,X_- &= 0\,,\\
(\mathrm{d}-H)\,\mathrm{Im}\,X_- &= c_0\;g_s*\lambda(F)\,.
\end{aligned}
\tag{5.3.1}
$$

Notice the presence of the constant c_0. Indeed, differently from the generalized calibrations, the source energy density is not minimized [30]. It is the combined bulk-brane energy density that is extremized.

Keeping only the parameters a that are non-zero in the de Sitter solution (5.2.47), it is easy to compute

$$\begin{aligned} \mathrm{d}(\mathrm{Im}\, X_-) &= [(a_0^r + a_{66}^r - a_{55}^r)[p(q_1+q_2)(e^1\wedge e^3 + e^2\wedge e^4) \\ &- (q_1-q_2)(e^1\wedge e^2 - e^3\wedge e^4)]\wedge e^5\wedge e^6 \\ &- (a_{11}^r + a_{44}^r - a_{22}^r - a_{33}^r)[p(q_1-q_2)(e^1\wedge e^3 + e^2\wedge e^4) \\ &- (q_1+q_2)(e^1\wedge e^2 - e^3\wedge e^4)]\wedge e^5\wedge e^6 \,, \end{aligned} \tag{5.3.2}$$

and

$$H\wedge \mathrm{Im}\, X_- = -2h\;(a_0^r + a_{66}^r - a_{55}^r)\; e^1\wedge e^2\wedge e^3\wedge e^4\wedge e^5\wedge e^6 \,. \tag{5.3.3}$$

In order to have $\mathrm{d}(\mathrm{Im}\, X_-)$ proportional to $g_s * F_2$, one must impose the relation

$$a_{11}^r + a_{44}^r - a_{22}^r - a_{33}^r = 0 \;. \tag{5.3.4}$$

Then, one has

$$\begin{aligned} \mathrm{d}(\mathrm{Im}\, X_-) &= -c_0\; g_s * F_2 \;, \\ H\wedge \mathrm{Im}\, X_- &= -2\gamma^2\; c_0\; g_s * F_0 \;, \end{aligned} \tag{5.3.5}$$

with

$$c_0 = \frac{a_0^r + a_{66}^r - a_{55}^r}{\gamma} = -g_s \frac{\tilde{T}_0}{\gamma(c_1+c_2)} \;. \tag{5.3.6}$$

To obtain the second equality, we used the explicit expression (5.2.47), (5.2.35) for the parameters a, while c_1 and c_2 are defined in (5.2.49). Also, using (5.2.47), it is easy to show that the constraint (5.3.4) reduces to

$$x_0 = 2\tilde{T}_0 - (T_{11} + T_{22} + T_{33} + T_{44}) = 0 \qquad \Leftrightarrow \qquad (2\gamma^2 - 1)\; h^2 = 0 \;. \tag{5.3.7}$$

Therefore, for[13]

$$\gamma^2 = \frac{1}{2} \tag{5.3.8}$$

we can write a differential equation for $\mathrm{Im}\, X_-$

$$(\mathrm{d} - H)\,\mathrm{Im}\, X_- = c_0\; g_s * \lambda(F) \,, \tag{5.3.9}$$

which is the non-supersymmetric analogue of the supersymmetry equations[14] for $\mathrm{Im}\,\Phi_-$.

Notice that the value $\gamma^2 = 1/2$ is compatible with the condition (5.2.31) for having de Sitter solutions. The value of the constant c_0 is also fixed by the solution. Indeed, in order for X_- to reproduce the correct Born-Infeld action (5.2.4), we have to impose $c_0\,\gamma = 1$. This relation is automatically satisfied for supersymmetric backgrounds, where $c_0 = \gamma = 1$ and the pullback of $\mathrm{Re}\,\Omega$ agrees with the DBI action on the solution. In non-supersymmetric backgrounds, the condition $c_0\,\gamma = 1$ plus (5.3.6)

[13]Clearly also $h = 0$ (no NSNS flux) is a solution to this constraint. It would be interesting to explore the possibility of having de Sitter or non-supersymmetric Minkowski solution with $h = 0$. Notice that, in this case, the condition of having $F_0 \neq 0$ [42], necessary to avoid de Sitter no-go theorems [70], is not required.

[14]Notice that from the equation for $\mathrm{Im}\, X_-$ we recover the condition $T_0 > 0$ (5.2.17). Indeed, as in [29], starting from (5.2.38) we have

$$\frac{T_0}{p+1}\int_M \mathrm{vol}_{(6)} = -\int_M \langle \mathrm{d}_H F, \mathrm{Im}\, X_-\rangle = -\int_M \langle F, \mathrm{d}_H\, \mathrm{Im}\, X_-\rangle = c_0\; g_s\; \int_M \langle *\lambda(F), F\rangle > 0 \;.$$

fix the value of the constant, $c_0 = \sqrt{2}$.

We can now show that the d_H - closure can be imposed on $\mathrm{Re}\, X_-$. Indeed, the three-form part of $\mathrm{Re}\, X_-$ can be written as

$$\begin{aligned} \mathrm{Re}\, X_3 \;\; = \;\; & -[b_0^r - Tr(b_{kj}^r) + b_{kj}^r(g^{kl}dx^j + g^{jl}dx^k)\iota_l]\, \mathrm{Re}\,\Omega \\ & +[b_0^i - Tr(b_{kj}^i) + b_{kj}^i(g^{kl}dx^j + g^{jl}dx^k)\iota_l]\, \mathrm{Im}\,\Omega \\ & +[(b_k^{i\,R} - b_k^{i\,L})\; dx^k + g^{kl}(b_k^{rR} - b_k^{rL})\; \iota_l J] \wedge J \ , \end{aligned} \tag{5.3.10}$$

where, as for $\mathrm{Im}\, X_3$, we have defined

$$\begin{aligned} b_0^r &= \mathrm{Im}(\tilde{\alpha}_0 - \alpha_0) & b_{kj}^r &= \mathrm{Im}(\tilde{\alpha}_{kj} - \alpha_{kj}) \ , \\ b_0^i &= \mathrm{Re}(\tilde{\alpha}_0 + \alpha_0) & b_{kj}^i &= \mathrm{Re}(\tilde{\alpha}_{kj} + \alpha_{kj}) \ , \\ b_k^{r\,L} &= \mathrm{Re}(\tilde{\alpha}_k^L + \alpha_k^L) & b_k^{r\,R} &= \mathrm{Re}(\alpha_k^R + \tilde{\alpha}_k^R) \ , \\ b_k^{i\,L} &= \mathrm{Im}(\tilde{\alpha}_k^L - \alpha_k^L) & b_k^{i\,R} &= \mathrm{Im}(\alpha_k^R - \tilde{\alpha}_k^R) \ . \end{aligned} \tag{5.3.11}$$

Consistently with (5.2.46), we can choose

$$\begin{aligned} & b_0^r = 0 \, , \\ & b_k^{r\,L} = b_k^{r\,L} = b_k^{i\,L} = b_k^{i\,R} = 0 \qquad \forall\ k = 1, \ldots 6 \\ & b_{jk}^r = 0 \qquad\qquad\qquad\qquad\quad\;\; \forall\ j,k = 1, \ldots 6 \\ & b_{jk}^i = 0 \qquad \text{for } (kj) \notin \{(kk),\ (14),\ (23),\ (41),\ (32)\} \, . \end{aligned} \tag{5.3.12}$$

Furthermore, choosing

$$\frac{b_{14}^i}{t_1} = -\frac{b_{23}^i}{t_2\tau_3^2} \quad , \quad \frac{b_{11}^i}{t_1} + \frac{b_{33}^i}{t_2\tau_3^2} - \frac{b_{22}^i}{t_2\tau_3^2\lambda} - \frac{b_{44}^i}{t_1\lambda} = 0 \ , \tag{5.3.13}$$

we obtain

$$\mathrm{d}_H(\mathrm{Re}\, X_3) = \sqrt{t_1 t_2 t_3}\ \tau_3\tau_6\ p(1-\lambda)\ \left(b_0^i + \frac{b_{66}^i}{t_3\tau_6^2} - \frac{b_{55}^i}{t_3} \right) (q_2\ e^1 \wedge e^4 + q_1\ e^2 \wedge e^3) \wedge e^5 \wedge e^6 \ , \tag{5.3.14}$$

which is zero either in the SUSY solution, or by further setting

$$b_0^i = -\frac{b_{66}^i}{t_3\tau_6^2} + \frac{b_{55}^i}{t_3} \, . \tag{5.3.15}$$

While these equations are derived in the vanishing warp factor and constant dilaton limit, their extension to the general case is natural[15]

$$\begin{aligned} & \mathrm{d}_H(e^{2A-\phi}\, \mathrm{Re}\, X_-) = 0 \, , \\ & \mathrm{d}_H(e^{4A-\phi}\, \mathrm{Im}\, X_-) = c_0 e^{4A} * \lambda(F) \, . \end{aligned} \tag{5.3.16}$$

Inspired by the supersymmetric case, it is natural to ask if and when it might be possible to find an even-form counterpart to (5.3.16), X_+ , so that the equations on X_- and X_+ (together with flux Bianchi identities) imply the e.o.m. However it is not yet clear to us what the correct generalization

[15] Just like Φ_-, X_- is globally defined, and both B-field and the dilaton are needed in order to define an isomorphism between such forms and the positive and negative helicity spin bundles $S^{\pm}(E)$ as discussed in section 2.3.3. The dilaton assures the correct transformation under $GL(6)$, making the (non-pure) spinor $e^{-\phi}e^{-B}\, X_-$ the natural variable for the first order equations (5.3.16), as Ψ_- is in (4.4.1).

of the notion of compatibility is, and what algebraic properties X_+ should satisfy. Hoping for a symmetry with the supersymmetric solutions (and the possibility of having a solution to some variational problem) one may construct X_+ satisfying

$$d_H(e^{3A-\phi}X_+) = 0\,. \tag{5.3.17}$$

In general the odd form X_- should receive contribution from both pure spinors, but in our solution we have chosen to "decouple" the even pure spinor completely. Assuming X_+ has an expansion similar to that of X_-, i.e. X_+ does not receive contributions from Ω, this amounts to finding a closed two-form on $\mathfrak{g}_{5.17}^{p,-p,\pm 1} \times S^1$. It is indeed not hard to construct such a form for our solution, since the symplectic form itself is closed, provided $\tau_2 = 0$ (even if $\lambda \neq 1$, see (4.4.45), and footnote 15 of section 4.4.3). Even if we do not take $\tau_2 = 0$, finding a conformally closed X_+ of this form is always possible, since the manifold is symplectic. A better understanding of such first order equations applicable to non-supersymmetric backgrounds is a work in progress.

Finally, we would like to stress that we do not consider the open string sector of our solution (we set $\mathcal{F} = 0$ but do not analyse the worldvolume scalars). Note though that this would be needed for D-brane sources, while in our model, we can tune parameters in order to have only two O-planes (see (5.2.49)). Nevertheless, in the case of a D-brane, this is a weak point of our proposal since we are not able to properly address the question of the stability of the supersymmetry-breaking branes. By solving the bulk equations of motion, we extremize the energy density of the brane plus bulk system, but we cannot be sure that the solution is a minimum for arbitrary values of the parameters. The problem is currently under study. Note though that an unstable brane may also be interesting for some inflation scenarios.

In the next section we will give a partial justification of the stability of our solution by analysing the four-dimensional effective potential.

5.4 Four-dimensional analysis

The search for de Sitter vacua, or for no-go theorems against their existence, has generally been performed from a four-dimensional point of view [69, 70, 71, 42, 72, 73, 74, 76], analysing the behaviour of the four dimensional effective potential with respect to its moduli dependence. In this section, we want to make contact with this approach and show that our solution has the good behaviour one expects to find for de Sitter vacua, as far as the volume and the dilaton are concerned.

5.4.1 Moduli and $4d$ Einstein frame

Let us consider the ten-dimensional action (5.2.6). As discussed in chapter 1, by Kaluza-Klein reduction on the internal manifold, we should obtain a four-dimensional effective action for the moduli. In particular, in addition to the kinetic terms, the four-dimensional action will contain a potential for the moduli fields. Their number and the way they enter the potential will depend on the peculiar features of the single model.

A de Sitter solution of the four-dimensional effective action will correspond to a positive valued minimum of the potential. Determining the minima of the potential is general rather difficult, since, a priori one should extremize along all the directions in the moduli space. This complicated problem is generally solved only by numerical analysis, because of the large number of variables. However, some information can be extracted by restricting the analysis to a subset of the moduli fields.

For whatever choice of the manifold on which the compactification is performed, we are always able to isolate two universal moduli: the internal volume and the four-dimensional dilaton. Their appearance in the effective potential at tree-level is also universal. We will then only focus on these two moduli. We define the internal volume as

$$\int_M \mathrm{d}^6x\sqrt{|g_6|} = \frac{L^6}{2} = \frac{L_0^6}{2}\rho^3 \ , \tag{5.4.1}$$

where the factor of $\frac{1}{2}$ is due to the orientifold and the vacuum value is $\rho = 1$. Defining the ten-dimensional dilaton fluctuation as $e^{-\tilde{\phi}} = g_s e^{-\phi}$, the four-dimensional dilaton is given by

$$\sigma = \rho^{\frac{3}{2}}\, e^{-\tilde{\phi}} \, . \tag{5.4.2}$$

Then reducing the action (5.2.6), we obtain the four-dimensional effective action for gravity, $4d$ dilaton and volume modulus in the string frame

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{|g_4|}\left[\frac{L^6}{2}e^{-2\phi}(R_4 + 4|\nabla\phi|^2) - 2\kappa^2 U\right] , \tag{5.4.3}$$

with $U(\rho, \sigma)$ the four-dimensional potential. To derive the explicit form of the potential, we need to determine how the internal Ricci scalar, fluxes and source terms scale with the volume. For R_6 and the fluxes this is easily computed

$$R_6 \to \rho^{-1}\, R_6 \, , \quad |H|^2 \to \rho^{-3}\, |H|^2 \, , \quad |F_k|^2 \to \rho^{-k}\, |F_k|^2 \, . \tag{5.4.4}$$

The source term requires some more attention. As shown in (5.2.38),

$$2\kappa^2 T_p \,\hat{*}\langle j, \mathrm{Im}\, X\rangle = \frac{[(\mathrm{d}F_2 - HF_0)\wedge \mathrm{Im}\, X_3]_{1\ldots 6}}{\sqrt{|g_6|}} \ . \tag{5.4.5}$$

The terms in $\mathrm{Im}\, X_3$ in (5.2.40) appearing with a_0, a_{jk} and $a_k^{(L,R)}$ scale differently with the volume. Let us denote them by X_0, X_Ω and X_J, respectively

$$\mathrm{Im}\, X_3 = X_0 + X_\Omega + X_J \ . \tag{5.4.6}$$

Their ρ dependence is determined by the scaling of the forms J and Ω

$$J \to \rho J \ , \ \Omega \to \rho^{\frac{3}{2}}\Omega \ , \tag{5.4.7}$$

and by the metric factors in the gamma matrices of (5.2.40)

$$X_0 \to \rho^{\frac{3}{2}} X_0 \ , \ X_\Omega \to \rho^{\frac{1}{2}} X_\Omega \ , \ X_J \to \rho X_J \ . \tag{5.4.8}$$

Then, the source term scales as

$$\frac{[(\mathrm{d}F_2 - HF_0)\wedge \mathrm{Im}\, X_3]_{1\ldots 6}}{\sqrt{|g_6|}} \to \rho^{-\frac{3}{2}}\left(b_0 + b_1\, \rho^{-1} + b_2\, \rho^{-\frac{1}{2}}\right) \ , \tag{5.4.9}$$

where

$$\begin{aligned} b_0 &= \frac{[(\mathrm{d}F_2 - HF_0)\wedge X_0]_{1\ldots 6}}{\sqrt{|g_6|}} \, , \\ b_1 &= \frac{[(\mathrm{d}F_2 - HF_0)\wedge X_\Omega]_{1\ldots 6}}{\sqrt{|g_6|}} \, , \\ b_2 &= \frac{[(\mathrm{d}F_2 - HF_0)\wedge X_J]_{1\ldots 6}}{\sqrt{|g_6|}} \, , \end{aligned} \tag{5.4.10}$$

are vacuum values. Then the four-dimensional potential for ρ and σ becomes

$$
\begin{aligned}
U &= \frac{1}{2\kappa^2}\int_M \mathrm{d}^6x\sqrt{|g_6|}[e^{-2\phi}(-R_6+\frac{1}{2}|H|^2)+\frac{1}{2}(|F_0|^2+|F_2|^2)-2\kappa^2 T_p\ e^{-\phi}\,\hat{*}\langle j,\mathrm{Im}\,X\rangle] \\
&= \frac{L_0^6}{4g_s^2\kappa^2}\,\sigma^2\,[(-\frac{R_6}{\rho}+\frac{|H|^2}{2\rho^3})-\frac{g_s}{\sigma}\,(b_0+\frac{b_1}{\rho}+\frac{b_2}{\sqrt{\rho}})+\frac{g_s^2\,\rho^3}{2\sigma^2}(|F_0|^2+\frac{|F_2|^2}{\rho^2})]\,. \qquad (5.4.11)
\end{aligned}
$$

Note that the terms in b_1 and b_2 are purely non-supersymmetric contributions of the source. They are due to the new metric dependence of the source action with respect to the supersymmetric case.

In order to correctly identify the cosmological constant, but also to perform the study of the moduli dependence, we need to go to the four-dimensional Einstein frame

$$
g_{\mu\nu\ E}=\sigma^2\ g_{\mu\nu}\,. \qquad (5.4.12)
$$

The four-dimensional Einstein-Hilbert term transforms as[16]

$$
\begin{aligned}
\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{|g_4|}\frac{L^6}{2}e^{-2\phi}R_4 &= \frac{L_0^6}{2g_s^2\ 2\kappa^2}\int \mathrm{d}^4x\sqrt{|g_4|}\ \sigma^2 R_4 \\
&= M_4^2\int \mathrm{d}^4x\sqrt{|g_{4E}|}R_{4E}\,,
\end{aligned}
$$

where we denote Einstein frame quantities by E, and we introduced $M_4^2=\frac{L_0^6}{2g_s^2\ 2\kappa^2}$, the squared four-dimensional Planck mass. Similarly, the four-dimensional potential in the Einstein frame becomes

$$
U_E=\sigma^{-4}\ U=4\kappa^4 M_4^4\frac{e^{4\phi}}{(\frac{L^6}{2})^2}\ U\ , \qquad (5.4.14)
$$

and we can write the Einstein frame action as

$$
S=M_4^2\int \mathrm{d}^4x\sqrt{|g_{4E}|}\left(R_{4E}+\mathrm{kin}-\frac{1}{M_4^2}U_E\right)\,. \qquad (5.4.15)
$$

The cosmological constant, (5.2.13), is then related to the vacuum value of the potential

$$
\Lambda=\frac{1}{2M_4^2}U_E|_0\ . \qquad (5.4.16)
$$

5.4.2 Extremization and stability

In order to find a solution, one should determine the minima of the potential. For our choice of moduli, ρ and σ, one has

$$
\begin{aligned}
\frac{\partial U_E}{\partial\sigma} &= -\frac{M_4^2}{\sigma^5}[2g_s^2\,(|F_0|^2\rho^3+|F_2|^2\rho)+2\sigma^2\,(-\frac{R_6}{\rho}+\frac{|H|^2}{2\rho^3})-3\sigma\ g_s(b_0+\frac{b_1}{\rho}+\frac{b_2}{\sqrt{\rho}})] \qquad (5.4.17)\\
\frac{\partial U_E}{\partial\rho} &= \frac{M_4^2}{\sigma^2}\ [(\frac{R_6}{\rho^2}-\frac{3|H|^2}{2\rho^4})+\frac{g_s}{\sigma}\ (\frac{b_1}{\rho^2}+\frac{b_2}{2\sqrt{\rho^3}})+\frac{g_s^2}{2\sigma^2}(3|F_0|^2\rho^2+|F_2|^2)]\,. \qquad (5.4.18)
\end{aligned}
$$

In our conventions, the extremization conditions are

$$
\frac{\partial U_E}{\partial\sigma}|_{\sigma=\rho=1}=0\quad,\quad\frac{\partial U_E}{\partial\rho}|_{\sigma=\rho=1}=0\ , \qquad (5.4.19)
$$

[16]Under a conformal rescaling of the four dimensional metric we have

$$
g_{\mu\nu}\to e^{2\lambda}g_{\mu\nu}\qquad\Rightarrow\qquad\sqrt{|g_4|}\to e^{4\lambda}\sqrt{|g_4|}\ ,\ R_4\to e^{-2\lambda}R_4\ . \qquad (5.4.13)
$$

where $\sigma = \rho = 1$ are the values of the moduli on the vacuum. Actually, the conditions (5.4.19) are equivalent to the ten-dimensional dilaton e.o.m. and the trace of internal Einstein equation. Combining the dilaton equation (5.2.17) and the trace of the internal Einstein equation, (5.2.16), we can write the six-dimensional Ricci scalar as

$$R_6 = \frac{3}{2}|H|^2 - \frac{g_s^2}{2}(3|F_0|^2 + |F_2|^2) - \frac{g_s}{2}(T_0 - T)\,, \tag{5.4.20}$$

where

$$\begin{aligned} T_0 - T &= 2\kappa^2 T_p \hat{*}\langle j, C^m_m\rangle = \frac{[(\mathrm{d}F_2 - HF_0)\wedge(X_J + 2X_\Omega)]_{1\ldots 6}}{\sqrt{|g_6|}} \\ &= 2b_1 + b_2\,. \end{aligned} \tag{5.4.21}$$

In the last line we used (5.4.10). With this expression for $T_0 - T$, it is immediate to verify that (5.4.20) is indeed equal to the $\partial_\rho U_E$ in (5.4.19). Similarly, one can see that using (5.4.18), (5.4.9), (5.2.38) and (5.4.19), the dilaton equation (5.2.17) reduces to $\partial_\sigma U_E$ in (5.4.19).

From the equivalence of the ten-dimensional equations and (5.4.19) we see that the ten-dimensional solution discussed in the previous sections does indeed satisfy the extremization conditions (5.4.19). The next step is to see whether such extremum correspond to a minimum of the potential and whether, furthermore, it is stable.

Let us consider (5.4.18) and discuss the ρ dependence of the potential. It is convenient to define the function

$$P(\rho^2) = \frac{\partial U_E}{\partial \rho}\frac{\sigma^2\rho^4}{M_4^2}\,. \tag{5.4.22}$$

It is easy to check that $P(\rho^2)$ is negative for $\rho = 0$ and positive for $\rho \to \infty$. Hence there must be a real positive root and this is a minimum of U_E. A priori, $P(\rho^2)$ could have other zeros. Let us focus only on the situation in which $b_2 = 0$, which, in particular, is the case for our ten-dimensional solution. In that case, $P(\rho^2)$ has two other roots which are either complex conjugate[17], or real and negative, according to the value of the parameters. Indeed, studying $\partial_{\rho^2}P$, one can show that $P(\rho^2)$ can be 0 only once. Therefore, at least for $b_2 = 0$, there is only one extremum of U_E in ρ and it is a minimum. So satisfying the extremization in ρ is enough for the stability.

Let us now analyze the σ dependence of (5.4.14). It is easy to see that the potential admits an extremum for

$$\sigma_\pm = \frac{1}{4a}\left(3b \pm \sqrt{8b^2\left(\frac{9}{8} - \frac{4ac}{b^2}\right)}\right)\,, \qquad \frac{4ac}{b^2} < \frac{9}{8}\,, \tag{5.4.23}$$

where for simplicity we introduced

$$\begin{aligned} a &= -R_6\rho^{-1} + \frac{1}{2}|H|^2\rho^{-3}\,, \\ b &= g_s(b_0 + b_1\rho^{-1} + b_2\rho^{-\frac{1}{2}})\,, \\ c &= \frac{g_s^2}{2}\rho^3(|F_0|^2 + |F_2|^2\rho^{-2})\,. \end{aligned} \tag{5.4.24}$$

In our case, asking for $\sigma = 1$ and using the extremization in σ in (5.4.19), which can be written as $2a - 3b + 4c = 0$, we find that the minimum in σ_- corresponds to

$$a - 2c < 0\,. \tag{5.4.25}$$

[17] Since the polynomial is real, they come in conjugate pairs.

This condition is satisfied by our solution choosing $\gamma^2 = \frac{1}{2}$, as we can see from (5.2.30). Therefore, our solution is at the minimum in σ, and it is then stable both in the volume and the dilaton moduli.

It is easy to see that the four-dimensional potential takes a positive value at the minimum, and, hence, the minimum corresponds to a de Sitter vacuum. In [71], it has been shown that the potential has a strictly positive minimum in σ for

$$1 < \frac{4ac}{b^2} < \frac{9}{8} \,, \tag{5.4.26}$$

where the lower bound comes from asking the potential to be never vanishing (strictly positive). This condition is satisfied by our solution.

In addition, we can actually compute the value of the potential at $\sigma = \rho = 1$. Starting from (5.4.14) and using the two equations of (5.4.19), we obtain

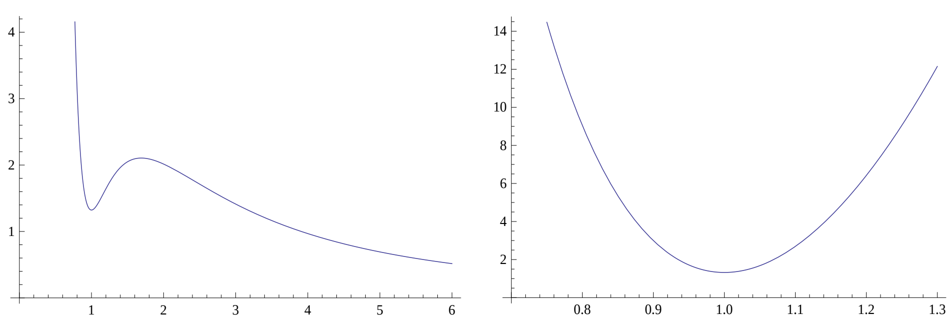
$$\frac{U_E}{M_4^2} = \frac{1}{3}\left(\frac{g_s}{2}(T_0 - T) + g_s^2|F_0|^2 - |H|^2\right) \,. \tag{5.4.27}$$

Using (5.2.14) and (5.4.20), one can show that the four-dimensional Ricci scalar is proportional to (5.4.27), $R_4 = 2U_E/M_4^2$. For $\gamma^2 = 1/2$, R_4 is positive (see the discussion below (5.2.31)), and hence so is the value of the potential at the minimum.

Note also that, for $\gamma^2 = 1/2$, the last two terms in (5.4.27) cancel each other and the entire contribution to the cosmological constant comes from sources, $(T_0 - T)$. For supersymmetry breaking branes, this contribution is never vanishing but, for generic situations, we do not know what its sign is. It would be nice to have a model independent argument to determine whether, for this mechanism of supersymmetry breaking, the resulting four-dimensional space is always de Sitter.

As a further check of the existence of a de Sitter minimum for our solution, we can plot the four-dimensional potential U_E as a function of σ and ρ for some values of the parameters

$$\begin{aligned} &t_1 = t_2 = t_3 = \tau_3 = \tau_6 = 1 \,, \\ &q_1 = 1 \,,\; q_2 = 3 \,,\; p = \frac{\cosh^{-1}(2)}{\pi} \,, \\ &\lambda = 5 \,,\; \gamma = \frac{1}{\sqrt{2}} \,,\; h = 4 \,. \end{aligned} \tag{5.4.28}$$



$\frac{1}{M_4^2}U_E(\sigma, \rho = 1)$ $\qquad$ $\frac{1}{M_4^2}U_E(\sigma = 1, \rho)$

Figure 5.1: Dependence of the potential on dilaton and volume modulus

Chapter 6

Conclusion

In thesis, we studied flux vacua of ten-dimensional type II supergravity, compactified on a solvmanifold (twisted tori). Finding such vacua is an important step in the process of trying to relate ten-dimensional string theory to low energy four-dimensional particle physics, as discussed in chapter 1.

When looking for solutions towards four-dimensional Minkowski space-time, preserving supersymmetry in the vacuum serves as a guiding line, and also offers technical advantages. As discussed in chapter 2, this requirement plays a key role: in absence of flux, it characterizes the internal manifold to be a Calabi-Yau (CY) manifold. Reductions over a CY are unfortunately not phenomenologically satisfying, because of the moduli problem. Allowing for internal fluxes in the vacuum helps to solve this problem, because they generate a potential which can fix some, if not all, the moduli. In that case, the internal manifold is no longer restricted to be a CY. The conditions to get a supersymmetric Minkowski flux vacuum can be reformulated in terms of Generalized Complex Geometry (GCG). Thanks to this mathematical formalism, the internal manifold is characterized to be a Generalized Calabi-Yau (GCY). This class of manifolds, larger than the CY, incorporates some solvmanifolds. This is why we used them for our compactifications.

In chapter 3 and associated appendix B, we gave an account on the properties of these manifolds. Thanks these properties, and to the reformulation of the conditions in terms of GCG, one can now look for supersymmetric vacuum by a direct resolution of the various constraints, instead of obtaining these solutions as duals of CY solutions. So we discussed a resolution method to get vacua on these manifolds, and then provided a list of known solutions. The method was then adapted to find new solutions of a particular kind: those with intermediate $SU(2)$ structure, meaning the two internal supersymmetry spinorial parameters are neither parallel nor orthogonal. Such solutions, even if covered by the GCG formalism, were at first not easy to find because of intricate orientifold projection conditions and supersymmetry equations. We introduced appropriate new variables, and the problem got simplified. Then, three new solutions were presented. By taking the limit where the spinors become either parallel or orthogonal, known solutions were recovered, and a new one was found. We also discussed how an intermediate $SU(2)$ structure solution could be related by a β-transform to an $SU(3)$ structure solution. We illustrated this situation with one of the solutions found.

In chapter 4, we discussed the idea of relating all supersymmetric Minkowski flux vacua on solvmanifolds by some transformation. As discussed in chapter 3, T-duality is known to relate some solutions on torus to solutions on twisted tori, but not all of them: some non-T-dual solutions were found using GCG tools. So we proposed a transformation named the twist, which can relate solutions on torus to solutions on solvmanifolds. The main building block of the transformation is a local $GL(d)$ matrix which constructs Maurer-Cartan one-forms of a solvmanifold out of those of a torus. Doing so, this operator is responsible for the topology change when going from one manifold to the other. This $GL(d)$ transformation is then embedded in a local $O(d,d)$ transformation. The latter acts equivalently on vectors and one-forms of the generalized tangent bundle of GCG. The transformation can then be extended by considering more general local $O(d,d)$ transformations. In addition to the topology change just discussed, one can act on the B-field by a B-transform, transform the metric with scalings, accordingly shift the dilaton, and finally allow for two $U(1)$ actions on the GCG pure spinors. With these actions, the whole NSNS sector gets transformed. The transformation of the RR sector is read indirectly via the supersymmetry conditions on the pure spinors. Note that on the contrary to T-duality, the transformed RR fluxes get contributions from both RR and NSNS sectors. Given this general twist transformation, we worked out general constraints in order to use it as a solution generating technique. We then solved these constraints to recover all known solutions on nilmanifolds, even those which are not T-dual to a torus solution. We also solved the constraints to construct a new solution on a solvmanifold. Finally, we discussed if the twist transformation could be used to get non-geometric solutions. As a side result, we presented another new solution in type IIA with an $O6$-plane on a solvmanifold. This solution is fully localized, on the contrary to previous solutions which had two intersecting smeared sources. In a final section, we considered flux solutions of heterotic string. In this context, two solutions are known on internal manifolds given locally by $K3 \times T^2$, while globally, one is a trivial fibration and the other is not. These two solutions are known to be related by the so-called Kähler/non-Kähler transition, a chain of dualities which involves a lift to M-theory. The twist transformation, designed to reproduce such a topology change, can be used to relate these two solutions. To do so, we first reformulated the conditions to get a supersymmetric solution of heterotic string in terms of GCG pure spinors. We were then able to act with the twist transformation. The introduction of GCG in this context lead to an interesting discussion, including the idea of a possible extension of the generalized tangent bundle to the gauge bundle. One could then act with a subset of $O(d+16, d+16)$ to transform the gauge fields. Details on this idea, and on the twist transformation are given in appendix C.

Motivated by recent cosmological observations, we finally discussed in chapter 5 ten-dimensional solutions towards four-dimensional de Sitter space-time. These solutions have a major disadvantage: they do not preserve supersymmetry. For supersymmetric Minkowski solutions, the supersymmetry conditions, together with the Bianchi identities for the fluxes, imply that equations of motion are automatically satisfied. This eases the resolution since equations of motion can be second order differential equations, while supersymmetry conditions are first order. Therefore, it is technically more difficult to get a de Sitter solution, as one goes back to solving the equations of motion. Another difficulty comes from obtaining a positive cosmological constant. In presence of supersymmetric sources and constant warp factor and dilaton, most of the supergravity classical contributions to the four-dimensional Ricci scalar come with a negative sign. One can then consider additional non-classical ingredients, or consider purely four-dimensional compactifications like non-geometric ones. Instead, we proposed an ansatz for non-supersymmetric sources. The main result of this proposal was to provide an additional positive contribution to the cosmological constant. We gave an explicit example of a de Sitter solution found with this ansatz on a solvmanifold. This solution can be understood as a deformation of the new supersymmetric Minkowski solution found via the twist transformation in chapter 4. A last requirement, which is always difficult to fully satisfy, is the four-dimensional stability of the solution. In our case, it was partially examined: the solution was found stable in the volume and dilaton moduli. The ansatz proposed for the sources mimics the supersymmetric situation. For the latter, the DBI action can be replaced by an action, over the whole internal manifold, given by a particular GCG pure spinor. Similarly in our proposal, the DBI action is replaced by a general expansion over a base given by the GCG pure spinors. This generality provided more freedom to solve the Einstein equations, and helped to get a positive cosmological constant. The GCG pure spinors are also involved in the supersymmetry conditions. Mimicking again the supersymmetric situation, we proposed a set of first order equations generalizing the supersymmetry conditions, so that they could imply, with the Bianchi identities, the equations of motion. These first order equations were only partially solved, and further study on such a proposal was left to future work.

Given all the flux vacua studied in this thesis, the next step would be to work out the effective actions over them, as discussed in chapter 1. To do so, one should first determine the light modes. This is actually an important unsolved problem. The answer is known in absence of fluxes: for a CY, the light modes are massless, and are given by harmonic forms. In presence of fluxes on a warped CY, one can still use this result when going to the large volume limit, where the fluctuations of the fluxes can be neglected. But on twisted tori, fluxes are responsible for the non-trivial topology so they have to be taken into account. Determining the light modes is much more involved. Generic formulas for four-dimensional actions, potentials, are known away from the CY: they are given in terms of internal integrals over expressions involving GCG pure spinors [83, 84]. Nevertheless, one does not know what is the basis of light modes on which to develop these expressions. Instead, consistent truncations are considered [85]: they correspond to truncations of the spectrum to a set of modes, which do not have to be the full set of lightest modes. But the truncation is consistent, in the sense that a four-dimensional solution obtained from the truncated and reduced action is liftable to a ten-dimensional solution. Most of the four-dimensional studies, in particular when looking for a de Sitter solution, are done in this set-up. A better understanding of these problematics is a first direction to pursue. It is necessary in order to make contact with phenomenology.

Another important aspect for phenomenology mentioned in chapter 1 is model building. Out of type II compactifications, the four-dimensional action is not enough to recover some supersymmetric extension of the standard model. In particular, in order to get non-abelian gauge groups, one adds additional ingredients like intersecting probe branes [86]. This kind of models has always been worked out on torus. Trying to work them out on twisted tori, where the fluxes needed for a four-dimensional scalar potential are present, could be interesting. Calibration conditions for sources on such backgrounds have been worked out, so such a model building is in principle doable. Similarly, recent advances in the F-theory approach should be studied, and some link with our GCG formalism would be interesting.

We would also like to come back to several questions raised by the study of flux compactifications. Flux backgrounds of heterotic string indicated similarities and differences with the type II set-up. The similarities are due to the S-duality relating type II at the orbifold point (type I) and heterotic string. Further comparison between the two set-ups could be interesting. In particular, we may learn more on the role of open string degrees of freedom, which are treated rather differently on the two sides of the duality. The role of the internal curvature, and the different sources on both sides, also need further study.

In heterotic compactifications, the H-flux appeared not to play the same role as in type II. For heterotic, it is not related to a twisting of the generalized tangent bundle. The H-flux is actually the S-dual of the RR fluxes for twisted tori solutions in type II. So the unusual role of the H-flux in heterotic is related to our poor understanding of RR fluxes in type II supergravity, where we do not have a clear geometric picture of the RR fluxes in terms of GCG. The RR fluxes bring constraints on solutions, which make the restriction of the internal manifold to be a GCY, not a complete characterization. A better understanding of the RR fluxes, in particular of the restriction they bring on the internal manifold, would be interesting. This may also shed light on heterotic flux compactifications.

At the world-sheet level, similar problems occur. Making the GCG formalism appear at the world-sheet level would be interesting. This formalism is successful in describing the NSNS sector, so GCG structures in sigma-models have already appeared for purely bosonic strings [24, 25, 26]. Trying to extend this within the Berkovits formalism could help to understand better the role of the RR fluxes.

Coming back to the work presented in this thesis, several points could lead to further study. One could try to make further use of the twist transformation. It is tempting to try implement the twist transformation at the world-sheet level, similarly to T-duality [87]. The mixing between NSNS and RR sector in the twist transformation would unfortunately require a non-trivial use of the Berkovits formalism, which makes such an implementation difficult. Not going that far, one could try to implement the twist transformation at the level of the supergravity action. The four-dimensional potentials are given in terms of the GCG pure spinors, and the twist transformation is acting on those. So we should be able to relate an action obtained by compactifying on a torus to an action obtained on a twisted tori. The transformation relating the two should correspond to some gauging, in the context of gauged supergravity, where one can relate four-dimensional theories corresponding to compactifications on different manifolds. To do so, one gauges some symmetries of the reduced action. In order to preserve supersymmetry in this procedure, the gauging has to fulfill some further constraints [88]. Similarly for us, we consider the local $O(d, d)$ group, and the transformation has to satisfy some constraints to get supersymmetric solution. These similarities, and their use at the level of the four-dimensional actions, should be further studied.

Our proposal for non-supersymmetric sources deserves further justification. In particular, one can ask how much the geometry of the subspace wrapped by the source can differ from the supersymmetric case. This problem is related to the question of calibration, and the stability of the source, that also need to be further studied. For instance one could look at known examples of non-BPS sources [89].

We also began to develop a first order formalism. This set of equations would imply, together with the Bianchi identities for the fluxes, the equations of motion. Differently from similar proposals [78], we try to preserve the closure of some polyform, which would correspond to some integrability condition. One should then understand what geometric structure would correspond to such an integrability.

Last but not least, we mentioned and illustrated how GCG can provide a nice understanding of non-geometric situations. Though, one still does not have a clear ten-dimensional description of non-geometric solutions. Such solutions are found at the four-dimensional level, and these can be of particular interest for de Sitter compactifications, because the non-geometric fluxes bring positive contributions to the cosmological constant. A better understanding of the ten-dimensional point of view would useful. Non-geometric compactifications have appeared through different other perspectives (see [90] and references therein), and drawing the link between the various approaches would also be interesting.

Hopefully, with all these ideas, we would go closer to extensions of the standard model of particle physics, but also get a better understanding of string theory.

Appendix A

Conventions for supersymmetric vacua

A.1 Differential forms

In this appendix we give our conventions on internal gamma matrices, differential forms, contractions, and the Hodge star.

We choose hermitian γ matrices (they are all purely imaginary and antisymmetric): $\gamma^{i\dagger} = \gamma^i$. Here are some identities used (see [91] for more):

$$\begin{aligned} \{\gamma^m, \gamma^n\} &= 2g^{mn} \qquad & [\gamma^m, \gamma^n] &= 2\gamma^{mn} \\ \{\gamma^{mn}, \gamma^p\} &= 2\gamma^{mnp} \qquad & [\gamma^{mn}, \gamma^p] &= -4\delta^{p[m}\gamma^{n]} \\ \{\gamma^{mnpq}, \gamma^r\} &= 2\gamma^{mnpqr} \qquad & [\gamma^{mnpq}, \gamma^r] &= -8\delta^{r[m}\gamma^{npq]} \ . \end{aligned} \tag{A.1.1}$$

We take as a convention for a p-form A:

$$\gamma^{\mu_1 \dots \mu_p} \leftrightarrow \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p} \ , \qquad A = \frac{1}{p!} A_{\mu_1 \dots \mu_p} \gamma^{\mu_1 \dots \mu_p} \ . \tag{A.1.2}$$

For a p-tensor A, we define the antisymmetrization as

$$A_{[\mu_1 \dots \mu_p]} = \frac{1}{p!} (A_{\mu_1 \mu_2 \mu_3 \dots \mu_p} - A_{\mu_2 \mu_1 \mu_3 \dots \mu_p} + A_{\mu_2 \mu_3 \mu_1 \dots \mu_p} + \dots + A_{\mu_3 \mu_4 \mu_1 \mu_2 \mu_5 \dots \mu_p} + \dots) \tag{A.1.3}$$

with the $p!$ possible terms on the right-hand side. For a p-form A and q-form B, we have the convention:

$$\frac{1}{(p+q)!} (A \wedge B)_{\mu_1 \dots \mu_{p+q}} = \frac{1}{p!q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}]} \ . \tag{A.1.4}$$

For a p-form A and a 1-form $b = b_\mu \gamma^\mu$, we define the contraction:

$$b \llcorner A = \frac{1}{p!} b^\nu A_{\mu_1 \dots \mu_p} \ p \ \delta_\nu^{[\mu_1} \gamma^{\mu_2 \dots \mu_p]} = \frac{1}{(p-1)!} b^{\mu_1} A_{\mu_1 \dots \mu_p} \gamma^{\mu_2 \dots \mu_p} \ . \tag{A.1.5}$$

It can also be noted with ι. For generic 1-form ξ, p-form A and q-form B, one has:

$$\xi \llcorner (A \wedge B) = (\xi \llcorner A) \wedge B + (-1)^p \ A \wedge (\xi \llcorner B) \ . \tag{A.1.6}$$

We now give the conventions for the Hodge star $*$, with a given metric g. We introduce the totally antisymmetric tensor ϵ by $\epsilon_{\mu_1 .. \mu_m} = +1/-1$ for $(\mu_1 .. \mu_m)$ being any even/odd permutation of $(1..m)$, and 0 otherwise. Then, the convention used for the Hodge star is[1]

$$*(\mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_k}) = \frac{\sqrt{|g|}}{(d-k)!} (-1)^{(d-k)k} \ \epsilon^{\mu_1 .. \mu_k \ \mu_{k+1} .. \mu_d} \ g_{\mu_{k+1} \nu_{k+1}} \dots g_{\mu_d \nu_d} \ \mathrm{d}x^{\nu_{k+1}} \wedge \dots \wedge \mathrm{d}x^{\nu_d} \ , \tag{A.1.7}$$

[1] We take the same "awkward sign convention" as in [29], and use the same pure spinors SUSY equations and the same calibration of the sources. See footnote 1 in chapter 2.

with d the dimension of the space, $|g|$ the determinant of the metric. In the basis $(\xi^1,...,\xi^d)$, with diagonalized metric D, we get for a k-form:

$$*(\xi^{\mu_1}\wedge...\wedge\xi^{\mu_k}) = (-1)^{(d-k)k}\ \frac{\epsilon_{\mu_1...\mu_d}}{\sqrt{|g|}} D_{\mu_{k+1}\mu_{k+1}}...D_{\mu_d\mu_d}\xi^{\mu_{k+1}}\wedge...\wedge\xi^{\mu_d}\ , \tag{A.1.8}$$

without any summation on μ_{k+1}, ..., μ_d, as we took off the $(d-k)!$, i.e. these indices are fixed; the $\epsilon_{\mu_1..\mu_d}$ is then only there for a sign. Note for a p-form A_p, one has:

$$**A_p = (-1)^{(d-p)p}\ A_p = (-1)^{(d-1)p}\ A_p\ . \tag{A.1.9}$$

A.2 Structure conditions and compatibility

A.2.1 $SU(2)$ structure conditions

In this appendix we derive in a specific way the $SU(2)$ structure conditions given in section 2.2.2. We start by considering a globally defined spinor η_+: this gives an $SU(3)$ structure which has the properties (2.2.14). Let us now assume there is some holomorphic globally defined one-form z, for which we recall $||z||^2 = \overline{z}\llcorner z = z\llcorner\overline{z} = \overline{z}^\mu z_\mu = 2$. One can then always define two-forms from it:

$$j = J - \frac{i}{2}z\wedge\overline{z} \qquad \omega = \frac{1}{2}\overline{z}\llcorner\Omega\ . \tag{A.2.1}$$

Note that j is clearly real. We are going to show that these define an $SU(2)$ structure (the one naturally embedded in the $SU(3)$) since they satisfy the conditions (2.2.16), (2.2.17), and (2.2.18).

Holomorphicity is defined with respect to an almost complex structure. Then, one can always have an hermitian metric (its non-zero components have one index holomorphic and the other one antiholomorphic). Using this metric and some holomorphicity arguments in six dimensions, we first get that $z\llcorner\Omega = 0$, $z\llcorner z = \overline{z}\llcorner\overline{z} = 0$. Furthermore, we get that ω is holomorphic, and deduce the following structure conditions:

$$\omega\wedge\omega = 0\ , \tag{A.2.2}$$

$$z\llcorner\omega = 0,\ \overline{z}\llcorner\omega = 0\ . \tag{A.2.3}$$

Using the same arguments, we get that $z\wedge\Omega = 0$, and using (A.1.6), we have: $0 = \overline{z}\llcorner(z\wedge\Omega) = 2\Omega - z\wedge(\overline{z}\llcorner\Omega)$, hence

$$\Omega = z\wedge\omega\ . \tag{A.2.4}$$

Let us now recover the structure conditions involving j. We get using (A.1.6): $z\llcorner(\frac{z\wedge\overline{z}}{2}) = -z$, $\overline{z}\llcorner(\frac{z\wedge\overline{z}}{2}) = \overline{z}$. We have (using our almost complex structure and real indices) $\overline{z}\llcorner J = i\overline{z}$, because

$$(\overline{z}\llcorner J)_\nu = \overline{z}^\mu J_{\mu\nu} = -J_{\nu\mu}\overline{z}^\mu = -J_\nu{}^\mu\ \overline{z}_\mu = -(-i)\overline{z}_\nu = i\overline{z}_\nu\ . \tag{A.2.5}$$

So we deduce from the definition of j the following structure conditions:

$$\overline{z}\llcorner j = 0,\ z\llcorner j = 0\ . \tag{A.2.6}$$

Using $J\wedge\Omega = 0$ and (A.2.4), we deduce $z\wedge j\wedge\omega = 0$, and using (A.1.6), we then get:

$$j\wedge\omega = 0\ . \tag{A.2.7}$$

To recover the remaining structure condition (2.2.16), we express the equality $\frac{4}{3}J^3 = i\Omega\wedge\overline{\Omega}$ in terms of z, j and ω, and get $\frac{4}{3}(j+\frac{i}{2}z\wedge\overline{z})^3 = iz\wedge\overline{z}\wedge\omega\wedge\overline{\omega}$. Then, using the previously derived properties, contracting last formula with z and then contracting with $\overline{z}$, we finally get:

$$2\ j^2 = \omega\wedge\overline{\omega}\ . \tag{A.2.8}$$

Going back to $\frac{4}{3}J^3 = i\Omega\wedge\overline{\Omega}$, one deduces with (A.2.8):

$$j^3 = 0\ . \tag{A.2.9}$$

A.2.2 Details on the compatibility conditions

In section 2.4.1, we explained that we needed a pair of compatible pure spinors. We mentioned that the compatibility conditions were actually implied by a set of $SU(2)$ structure conditions seen in section 2.2.2. We are going to prove this implication here. The $SU(2)$ structure conditions involved are (A.2.2), (A.2.7), (A.2.8), and (A.2.9). We will use the formulas (2.4.9) for the pure spinors, which are valid for any structure (intermediate or orthogonal $SU(2)$, $SU(3)$), hence this result is valid for any structure. We give the following useful formula for any p-form A_p and q-form B_q:

$$\lambda(A_p \wedge B_q) = (-1)^{pq}\lambda(A_p) \wedge \lambda(B_q) \ , \tag{A.2.10}$$

and we recall the compatibility conditions given in section 2.4.1

$$\left\langle \Phi_1, \overline{\Phi}_1 \right\rangle = \left\langle \Phi_2, \overline{\Phi}_2 \right\rangle \neq 0 \ , \tag{A.2.11}$$

$$\langle \Phi_1, V \cdot \Phi_2 \rangle = \left\langle \overline{\Phi}_1, V \cdot \Phi_2 \right\rangle = 0, \ \forall \ V = (v, \xi) \ \in \ TM \oplus T^*M \ . \tag{A.2.12}$$

In the following, we will use the Φ_i defined in (2.4.21) for IIA, but note these conditions are actually independent of the theory, since they are only involving a generic pair of pure spinors.

Using (2.4.9) for the pure spinors, the first compatibility condition gives

$$z \wedge \overline{z} \wedge \left(2k_\perp^2 j^2 + k_{||}^2 \omega \wedge \overline{\omega} - 2k_{||}k_\perp j \wedge \mathrm{Re}(\omega)\right) \neq 0 \ , \tag{A.2.13}$$

$$k_{||}^2 \frac{4}{3} i j^3 + i k_{||} k_\perp j^2 \wedge \mathrm{Re}(\omega) = 4\frac{z \wedge \overline{z}}{||z||^2} \wedge \left(j^2 (k_{||}^2 - k_\perp^2) + \frac{1}{2}\omega \wedge \overline{\omega}(k_\perp^2 - k_{||}^2) + 2j \wedge \mathrm{Re}(\omega) k_{||} k_\perp \right) \ . \tag{A.2.14}$$

One can see that imposing (A.2.7), (A.2.8) and (A.2.9), (A.2.14) is automatically satisfied. Only (A.2.13) remains to be satisfied; it corresponds to the volume form being non-zero.

Let us now focus on the second compatibility condition. Since this condition is valid for any V, it is sufficient to study it in the two different cases where $V = (v, 0)$ and $V = (0, \xi)$. Then let us first look at $V = (0, \xi)$ and the condition $\langle \Phi_1, V \cdot \Phi_2 \rangle = 0$. One gets:

$$\xi \wedge z \wedge \omega \wedge (k_{||} k_\perp \omega + (k_{||}^2 - k_\perp^2) j) = 0 \ . \tag{A.2.15}$$

As (A.2.15) is valid for any ξ, we get:

$$z \wedge \omega \wedge (k_{||} k_\perp \omega + (k_{||}^2 - k_\perp^2) j) = 0 \ . \tag{A.2.16}$$

If one imposes (A.2.2) and (A.2.7), (A.2.15) is automatically satisfied.

Let us now consider $V = (v, 0)$ and still $\langle \Phi_1, V \cdot \Phi_2 \rangle = 0$. Using (A.1.6) and the following useful formula valid $\forall v \ \in \ TM$, $\forall n \ \epsilon \ \mathbb{N}^*$

$$v \llcorner j^n = n \ j^{(n-1)} \wedge (v \llcorner j) \ , \tag{A.2.17}$$

one gets the following top form in terms of $v \llcorner z$, $v \llcorner j$, and $v \llcorner \omega$:

$$(v \llcorner z) \left(\frac{i}{2}\omega \wedge j^2 + \frac{z \wedge \overline{z}}{2} \omega \wedge (-k_{||} k_\perp \omega + j(k_\perp^2 - k_{||}^2)) \right) - z \wedge \left(i k_\perp^2 \omega \wedge j \wedge (v \llcorner j) + \frac{k_{||}^2}{2} j^2 \wedge (v \llcorner \omega) \right) \ .$$

Apart from the term in $j^2 \wedge (v \llcorner \omega)$, the previous expression is obviously zero when one imposes (A.2.2) and (A.2.7). Using (A.1.6) and (A.2.17), one has

$$v \llcorner (j^2 \wedge \omega) = 2 \ j \wedge v \llcorner (j) \wedge \omega + j^2 \wedge (v \llcorner \omega) \ . \tag{A.2.18}$$

Hence the term in $j^2 \wedge (v \llcorner \omega)$ is also zero when using (A.2.7), so the whole expression vanishes with (A.2.2) and (A.2.7). Thus, $\langle \Phi_1, V \cdot \Phi_2 \rangle = 0$ is automatically satisfied for any V when (A.2.2) and (A.2.7) are imposed.

One can play the same game with the condition $\left\langle \overline{\Phi}_1, V \cdot \Phi_2 \right\rangle = 0$. For $V = (0, \xi)$, one gets:

$$\xi \wedge z \wedge \left(k_{||} k_{\perp} (\omega \wedge \overline{\omega} - 2\ j^2) + j \wedge (k_{||}^2 \omega - k_{\perp}^2 \overline{\omega}) \right) = 0\ , \tag{A.2.19}$$

which is obviously satisfied by imposing (A.2.7) and (A.2.8). For the $V = (v, 0)$ case, one gets:

$$(v \llcorner z) \left(-\frac{4i}{3} k_{||} k_{\perp} j^3 + \frac{i}{2} j^2 \wedge (k_{||}^2 \omega - k_{\perp}^2 \overline{\omega}) - \frac{z \wedge \overline{z}}{2} \wedge (k_{||} k_{\perp} (\omega \wedge \overline{\omega} - 2 j^2) + j \wedge (k_{||}^2 \omega - k_{\perp}^2 \overline{\omega})) \right)$$
$$+ z \wedge \left(i k_{\perp} (v \llcorner j) \wedge j \wedge (2 k_{||} j + k_{\perp} \overline{\omega}) - \frac{k_{||}^2}{2} j^2 \wedge (v \llcorner \omega) \right) = 0\ . \tag{A.2.20}$$

Using the same kind of tricks as before ((A.2.9) gives $j^2 \wedge (v \llcorner j) = 0$), we get that (A.2.7), (A.2.8) and (A.2.9) imply that the whole expression is zero. Thus, $\left\langle \overline{\Phi}_1, V \cdot \Phi_2 \right\rangle = 0$ is automatically satisfied for any V when (A.2.7), (A.2.8) and (A.2.9) are imposed.

Appendix B

Intermediate $SU(2)$ structure solutions and solvmanifolds

In this appendix, we first provide a detailed discussion of the algebraic aspects and the compactness properties of nil- and solvmanifolds. Their geometric properties are discussed in section 3.2. Then, we focus on the variables introduced in section 3.4.1 for intermediate $SU(2)$ structure solutions: the projection basis variables. We rewrite the different equations to be solved in terms of these variables. Finally, we discuss the normalisation (3.3.6) out of the calibration of supersymmetric sources.

B.1 Solvable groups and compactness properties

B.1.1 Algebraic aspects of solvable groups

First algebraic definitions

We consider a connected and simply-connected real Lie group G of identity element e. H, N and Γ will be subgroups of G. We denote the associated Lie (sub)algebras of G, H, N by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{n}$. Connected and simply-connected (sub)groups are in one-to-one correspondence with the corresponding (sub)algebras. Many properties of the (sub)algebras will have their counterpart in the (sub)groups and vice versa.

The ascending series $(G_k)_{k\in\mathbb{N}}$, the descending series $(G^k)_{k\in\mathbb{N}}$ and the derived series $(D^kG)_{k\in\mathbb{N}}$ of subgroups of G are defined as

$$\begin{aligned} &G_0 = \{e\} \ , \ G^0 = D^0G = G \ , \\ &G_k = \{g \in G | [g, G] \subset G_{k-1}\} \ , \ G^k = [G, G^{k-1}] \ , \ D^kG = [D^{k-1}G, D^{k-1}G] \ , \end{aligned}$$

where the commutator of two group elements g and h is $[g,h] = ghg^{-1}h^{-1}$. We define in the same way the ascending, descending and derived series of $\mathfrak{g}$ or its subalgebras, by using the Lie bracket instead of the commutator, and 0 instead of e.

G is *nilpotent* respectively *solvable* if there exist k such that $G^k = \{e\}$ respectively $D^kG = \{e\}$. We define the same notions for the algebra $\mathfrak{g}$ replacing 0 with e. Lie (sub)algebras corresponding to nilpotent/solvable groups are nilpotent and solvable, respectively. The converse is also true. All nilpotent Lie algebras/groups are solvable (the converse is not true).

An ideal $\mathfrak{i}$ of $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ stable under the Lie bracket: $[\mathfrak{g}, \mathfrak{i}] \subset \mathfrak{i}$. Obviously $\mathfrak{i}$ is also a subalgebra. The subalgebras given in the previously defined series are all ideals.

The nilradical $\mathfrak{n}$ of the algebra $\mathfrak{g}$ is the biggest nilpotent ideal of $\mathfrak{g}$. The nilradical is unique [39, 92] as will be the corresponding subgroup N of G, also named nilradical.

To ideals of $\mathfrak{g}$ will correspond normal subgroups of G. We recall that a subgroup N is said normal if $\forall g \in G$, $gNg^{-1} \subset N$, i.e. it is invariant under conjugation (inner automorphisms). This property is necessary in order to be able to define a group structure on the quotient G/N. Note that the nilradical N of a solvable Lie group G as well as the subgroups D^kG of the derived serie are normal subgroups.

The adjoint action

Let V be a vector space over a field $\mathbb{K}$ and let $\mathfrak{g}$ be a Lie algebra over the same field. A representation of $\mathfrak{g}$ is a map $\pi : \mathfrak{g} \to \mathrm{End}(V)$ such that:

1. π is linear ;
2. $\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$.

There is a natural representation of a Lie algebra over itself called the adjoint representation:

$$\begin{aligned} ad \ &: \ \mathfrak{g} \to \mathrm{End}(|\mathfrak{g}|) \\ &\quad X \mapsto ad(X) = ad_X \ , \end{aligned}$$

where $|\mathfrak{g}|$ means the underlying vector space of the Lie algebra $\mathfrak{g}$, $\mathrm{End}(|\mathfrak{g}|)$ the space of all linear maps on it[1], and

$$\begin{aligned} \text{for } X \in \mathfrak{g} \ , \ ad_X \ &: \ \mathfrak{g} \to \mathfrak{g} \\ &\quad Y \mapsto ad_X(Y) = [X, Y] \ . \end{aligned}$$

We can obtain a matrix form of the adjoint representation from the structure constants in a certain basis of the Lie algebra. Let $\{E_a\}_{a=1,\dots,d}$ be a basis of a Lie algebra $\mathfrak{g}$, and the structure constants in that basis given by

$$[E_b, E_c] = f^a{}_{bc} E_a \ . \tag{B.1.1}$$

Then the matrices (a is the row index and c is the column index)

$$(M_b)^a{}_c = f^a{}_{bc} \tag{B.1.2}$$

provide a representation of the Lie algebra $\mathfrak{g}$.

A unimodular algebra $\mathfrak{g}$ is such that $\forall X \in \mathfrak{g}$, $tr(ad_X) = 0$. In view of what has been discussed, this is equivalent to $\sum_a f^a{}_{ba} = 0$, $\forall b$.

Let G be a Lie group and let V be a (real) vector space. A representation of G in V is a map $\pi : G \to \mathrm{Aut}(V)$ such that:

1. $\pi(e) = Id$;
2. $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$, $\forall\, g_1, g_2 \in G$.

There is a natural representation of the group over its algebra called the adjoint representation:

$$\begin{aligned} Ad \ &: \ G \to \mathrm{Aut}(\mathfrak{g}) \\ &\quad g \mapsto Ad(g) = Ad_g \ , \end{aligned}$$

where $Ad_g = exp^{Aut(|\mathfrak{g}|)}(ad_{X_g})$ for $X_g \in \mathfrak{g}$, $exp^G(X_g) = g$. Actually one can show the following relations between the representations:

[1] These maps do not necessarily respect the Lie bracket, or in other words, are not necessarily algebra morphisms. In particular, for $X \in \mathfrak{g}$, ad_X is not an algebra morphism.

$$\begin{tikzcd} G \arrow[r, "Ad"] & \mathrm{Aut}(\mathfrak{g}) \\ \mathfrak{g} \arrow[u, "exp^G"] \arrow[r, "ad"] & \mathrm{End}(|\mathfrak{g}|) \arrow[u, "exp^{Aut(|\mathfrak{g}|)}"'] \end{tikzcd}$$

The map ad then turns out to be the derivation[2] of Ad. At the level of the single elements, they act according to the following diagram:

$$\begin{tikzcd} g \arrow[r, "Ad"] & Ad(g) = Ad_g \\ X_g \arrow[u] \arrow[r, "ad"] & ad(X_g) = ad_{X_g} \arrow[u] \end{tikzcd}$$

One can show as well that the derivation of the inner automorphism I_g for $g \in G$ (the conjugation) is actually the adjoint action Ad_g:

$$d(I_g) = Ad_g \ . \tag{B.1.3}$$

Furthermore, for $\varphi : G \to G$ an automorphism, the following diagram is commutative:

$$\begin{tikzcd} G \arrow[r, "\varphi"] & G \\ \mathfrak{g} \arrow[u, "exp^G"] \arrow[r, "d\varphi"] & \mathfrak{g} \arrow[u, "exp^G"'] \end{tikzcd}$$

A Lie group is said to be exponential (the case for us) if the exponential map is a diffeomorphism. Denoting its inverse as log^G, then we deduce

$$I_g = exp^G \circ Ad_g \circ log^G \ . \tag{B.1.4}$$

Semidirect products

Most of the solvable groups we are interested in are semidirect products, we recall here some definitions.

Let us consider two groups H and N and a (smooth) action $\mu : H \times N \to N$ by (Lie) automorphisms. The semidirect product of H and N is the group noted $H \ltimes_\mu N$, whose underlying set is $H \times N$ and the product is defined as

$$(h_{i=1,2}, n_{i=1,2}) \in H \times N \ , (h_1, n_1) \cdot (h_2, n_2) = (h_1 \cdot h_2, n_1 \cdot \mu_{h_1}(n_2)) \ . \tag{B.1.5}$$

The semidirect product of Lie algebras can be defined in a similar way. Let $\mathfrak{d}(\mathfrak{h})$ be the derivation algebra of an algebra $\mathfrak{h}$ (for instance $ad \in \mathfrak{d}(\mathfrak{g})$). Let $\sigma : \mathfrak{g} \to \mathfrak{d}(\mathfrak{h})$, $X \mapsto \sigma_X$ be a representation of the Lie algebra $\mathfrak{g}$ in $|\mathfrak{h}|$. Then we can define the semidirect product $\mathfrak{g} \ltimes_\sigma \mathfrak{h}$ of the two Lie algebras with respect to σ in the following way:

- the vector space is $|\mathfrak{g}| \times |\mathfrak{h}|$
- the Lie bracket is $[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2]_\mathfrak{g}, [Y_1, Y_2]_\mathfrak{h} + \sigma_{X_1}(Y_2) - \sigma_{X_2}(Y_1))$.

This provides a Lie algebra structure to the vector space $|\mathfrak{g}| \times |\mathfrak{h}|$. Note that the fact σ is a derivation is important to verify the Jacobi identity for the new bracket.

If we denote $\mathfrak{g}' = \mathfrak{g} \times \{0\}$ and $\mathfrak{h}' = \{0\} \times \mathfrak{h}$ then $\mathfrak{h}'$ is an ideal of the new algebra and $\mathfrak{g}'$ is a subalgebra of it. Furthermore

$$\mathfrak{g}' + \mathfrak{h}' = \mathfrak{g} \ltimes_\sigma \mathfrak{h} \ , \ \mathfrak{g}' \cap \mathfrak{h}' = 0 \ . \tag{B.1.6}$$

[2]It is the derivative with respect to the parameters of the group element g, taken at the identity.

There is a unique decomposition of an element of $|\mathfrak{g}| \times |\mathfrak{h}|$ as a sum of an element of $|\mathfrak{g}|$ and one of $|\mathfrak{h}|$, thus we can think of it as the couple in $|\mathfrak{g}| \times |\mathfrak{h}|$ or as an element of a direct sum of vector spaces.

Let us consider a Lie group G and two subgroups H and N with N normal. If every element of G can be uniquely written as a product of an element in H and one in N, then one can show that $G \approx H \ltimes_\mu N$ with μ being the conjugation[3]. This point of view will be important for us. As discussed previously, the conjugation can be given in terms of the restriction of the adjoint action of H over $\mathfrak{n}$ as in (B.1.4), so we are able to determine μ in terms of $Ad_H(N)$. For exponential groups, as we consider here, the corresponding Lie algebra of $G = H \ltimes_\mu N$ is then clearly $\mathfrak{g} = \mathfrak{h} \ltimes_{ad_{\mathfrak{h}}(\mathfrak{n})} \mathfrak{n}$ (we just write ad in the following for simplicity).

Let us now consider a group G with a normal subgroup N of codimension 1. The Lie algebra $\mathfrak{g}$ has two components, $\mathbb{R}$ and $\mathfrak{n}$. We want to show that $\mathfrak{g}$ is isomorphic to $\mathbb{R} \ltimes_{ad} \mathfrak{n}$, and then, as discussed, we get that $G \approx \mathbb{R} \ltimes_\mu N$ with μ the conjugation. At level of the algebra, in terms of vector spaces, the isomorphism is obviously true. What needs to be verified is that the Lie brackets coincide. The Lie bracket of two elements of $\mathbb{R}$ or of $\mathfrak{n}$ clearly coincide with those of the corresponding two elements of $\mathbb{R} \ltimes_{ad} \mathfrak{n}$. Let us now take $X \in \mathbb{R},\ Y \in \mathfrak{n}$. We have for $\mathbb{R} \ltimes_{ad} \mathfrak{n}$:

$$[(X,0),(0,Y)] = (0, 0 + ad_X(Y) - ad_0(0)) = (0, [X,Y])\,, \tag{B.1.7}$$

which clearly coincides with the bracket $[X,Y]$ for $\mathfrak{g}$. We can conclude that $\mathfrak{g}$ is isomorphic to $\mathbb{R} \ltimes_{ad} \mathfrak{n}$ and thus the group is isomorphic to $\mathbb{R} \ltimes_\mu N$.

Solvable groups

According to Levi's decomposition, any real finite dimensional Lie algebra is the semidirect sum of its largest solvable ideal called the radical, and a semi-simple subalgebra. So solvable and nilpotent algebras do not enter the usual Cartan classification. Solvable algebras $\mathfrak{g}$ are classified with respect to the dimension of their nilradical $\mathfrak{n}$. One can show [63, 37] that $\dim\, \mathfrak{n} \geq \frac{1}{2}\dim\, \mathfrak{g}$. Since we are interested in six dimensional manifolds we will consider $\dim\, \mathfrak{n} = 3, \ldots, 6$. If $\dim\, \mathfrak{n} = 6$, $\mathfrak{n} = \mathfrak{g}$ and the algebra is nilpotent (they clearly are a subset of the solvable ones). There are 34 (isomorphism) classes of six-dimensional nilpotent algebras (see for instance [29, 93] for a list), among which 24 are indecomposable. Among the 10 decomposable algebras, there is of course the abelian one, $\mathbb{R}^6$. There are 100 indecomposable solvable algebras with $\dim\, \mathfrak{n} = 5$ (99 were found in [94], and [64] added 1, see [40] for a complete and corrected list), and 40 indecomposable solvable algebras with $\dim\, \mathfrak{n} = 4$ [64]. Finally, those with $\dim\, \mathfrak{n} = 3$ are decomposable into sums of two solvable algebras. There are only 2 of them, see Corollary 1 of [95]. In total, there are 164 indecomposable six-dimensional solvable algebras. For a list of six-dimensional indecomposable unimodular[4] solvable algebras, see [37].

Most of the solvable groups are semidirect products. For G a solvable group and N its nilradical, we consider the following definitions:

- If $G = \mathbb{R} \ltimes_\mu N$, G is called almost nilpotent. All three and four-dimensional solvable groups are of that kind [37].
- If furthermore, the nilradical is abelian (i.e. $N = \mathbb{R}^k$), G is called almost abelian.

The result at the end of the previous section applies here: any solvable group for which $\dim\, N = \dim\, G - 1$ is almost nilpotent. In fact N is a normal subgroup of G. Let us label the $\mathbb{R}$ direction with a parameter t, which we can take as a coordinate, with the corresponding algebra element being ∂_t. According to (B.1.4), we then have

$$\mu(t) = exp^N \circ Ad_{e^{t\partial_t}}(\mathfrak{n}) \circ log^N\ ,\ \ Ad_{e^{t\partial_t}}(\mathfrak{n}) = e^{ad_{t\partial_t}(\mathfrak{n})} = e^{t\ ad_{\partial_t}(\mathfrak{n})}\ . \tag{B.1.8}$$

[3] In particular it is the case for a group $G = H \ltimes_\nu N$ with ν being not the conjugation.

[4] See below equation (B.1.2) for a definition.

Furthermore, for the almost abelian case, we can identify N and $\mathfrak{n}$, so the *exp* and *log* correspond to the identity. Then, we obtain the simpler formula

$$\mu(t) = Ad_{e^{t\partial_t}}(\mathfrak{n}) = e^{t\ ad_{\partial_t}(\mathfrak{n})} \ . \tag{B.1.9}$$

We will mainly focus on solvable algebras with dim $\mathfrak{n} = 5$ (to which correspond almost nilpotent solvable groups) because, as we will discuss further, the compactness question is simpler to deal with.

B.1.2 Compactness

Existence of a lattice

We recall here that according to the definition[5] we adopt (section 3.2) a solvmanifold is a compact homogeneous space G/Γ obtained by the quotient of a connected, simply-connected solvable group and a discrete cocompact subgroup Γ, the lattice [37, 36]. The main result concerning the geometry of these manifolds is the Mostow bundle, and we refer to section 3.2 for its discussion (see in particular diagram (3.2.6) and [41] for the original reference). In this appendix, we come back to the problem of the existence of a lattice.

Whether a lattice exists or not, and so whether the manifold can be made compact is not always an easy question for non-nilpotent solvable groups. There is a simple necessary condition for a manifold to be compact, namely that the algebra has to be unimodular. Sufficient conditions are on the contrary more difficult to establish.

A theorem by Malcev [34] states that a connected and simply-connected nilpotent Lie group G admits a lattice if and only if there exists a basis for the Lie algebra $\mathfrak{g}$ such that the structure constants are rational numbers. This condition is always satisfied for all the 34 classes of nilpotent six dimensional algebras. For the non-nilpotent cases, several criteria have been proposed. The first is due to Auslander [39]. Despite its generality the criterion is difficult to use in concrete situations and we will not refer to it in our search for lattices. Details about it can be found in the original paper [39] and in [37]. Another criterion, which is closer to the one we use here, is due to Saitô [96]. It is less general than Auslander's because it applies to solvable groups that are algebraic subgroups of $GL(n, \mathbb{R})$ for some n. The criterion deals with the adjoint action of the group G over the nilradical $\mathfrak{n}$ of its algebra $\mathfrak{g}$. For an illustration, see [29].

The criterion we adopt in this work follows [37] and it applies to almost abelian solvable groups. As discussed above almost abelian solvable groups are characterized by the map $\mu(t)$ (B.1.9). Then the criterion states the group G admits a lattice if and only if it exists a $t_0 \neq 0$ for which $\mu(t_0)$ can be conjugated to an integer matrix. This criterion is very useful in practice since we have a simple formula (B.1.9) for $\mu(t)$.

In [37], some almost nilpotent (not almost abelian) cases were also proved to admit a lattice, thanks to some further technique that we will not consider here.

In section 3.2 we applied the compactness criterion mentioned above to the two algebras ε_2 and $\varepsilon_{1,1}$ (corresponding to $\mathfrak{g}^0_{3.5}$ and $\mathfrak{g}^{-1}_{3.4}$ given in the Table 1, respectively). Here we will review the argument for $\varepsilon_{1,1}$, using a change of basis closer to [37]. The algebra $\varepsilon_{1,1}$ is defined by

$$[E_1, E_3] = E_1 \ , \ [E_2, E_3] = -E_2 \ . \tag{B.1.10}$$

We have $\mathfrak{n} = \{E_1, E_2\}$ and $\partial_t = E_3$. Then, in the (E_1, E_2) basis,

$$ad_{\partial_t}(\mathfrak{n}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ \mu(t) = e^{t\ ad_{\partial_t}(\mathfrak{n})} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} \ . \tag{B.1.11}$$

[5] Let us emphasize the non-trivial result that, according to our (restrictive) definition solvmanifolds, these are always parallelizable (see [36] for a proof).

It is not possible to have $\mu(t_0)$ being an integer matrix for $t_0 \neq 0$. To check if the group admits a lattice, we have to find another basis where the matrix $\mu(t_0)$ can be integer. Let us consider the particular change of basis given by

$$P = \begin{pmatrix} 1 & c \\ 1 & \frac{1}{c} \end{pmatrix} \; , \; P^{-1} = \frac{1}{c - \frac{1}{c}} \begin{pmatrix} -\frac{1}{c} & c \\ 1 & -1 \end{pmatrix} \; , \tag{B.1.12}$$

where $c = e^{-t_1}$ and $t_1 \neq 0$. Then:

$$\hat{\mu}(t) = P^{-1} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} P = \begin{pmatrix} \frac{\sinh(t_1 - t)}{s_1} & -\frac{\sinh(t)}{s_1} \\ \frac{\sinh(t)}{s_1} & \cosh(t) + c_1 \frac{\sinh(t)}{s_1} \end{pmatrix} \; , \tag{B.1.13}$$

with $s_1 = \sinh(t_1)$ and $c_1 = \cosh(t_1)$. For $t = t_1$, we get

$$\hat{\mu}(t = t_1) = \begin{pmatrix} 0 & -1 \\ 1 & 2c_1 \end{pmatrix} \; . \tag{B.1.14}$$

The conjugated matrix $\hat{\mu}(t)$ can have integer entries for some non-zero $t = t_1$ when $2\cosh(t_1)$ is integer. In [37], $2\cosh(t_1) = 3$.

Let us now describe an example for which there is no lattice. We consider the algebra $\mathfrak{g}_{4.2}^{-p}$

$$[E_1, E_4] = -pE_1 \; , \; [E_2, E_4] = E_2 \; , \; [E_3, E_4] = E_2 + E_3 \; , \; p \neq 0 \; . \tag{B.1.15}$$

It is easy to check that the algebra is unimodular only for $p = 2$. This is a necessary condition for compactness, we can exclude all other values of p.

We have $\mathfrak{n} = \{E_1, E_2, E_3\}$ and $\partial_t = E_4$ (the algebra is almost abelian). Then, in the (E_1, E_2, E_3) basis,

$$ad_{\partial_t}(\mathfrak{n}) = \begin{pmatrix} p & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \; , \; \mu(t) = e^{t \; ad_{\partial_t}(\mathfrak{n})} = \begin{pmatrix} e^{pt} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & -te^{-t} & e^{-t} \end{pmatrix} \; . \tag{B.1.16}$$

Following [37], we are going to prove that this matrix cannot be conjugated to an integer matrix[6] except for $t = 0$. A way to verify if the matrix $\mu(t)$ can be conjugated to an integer one is to look at the coefficients of its characteristic polynomial $P(\lambda)$. This is independent of the basis in which it is computed, and hence, for the criterion to be satisfied it should have integer coefficients. Here we have:

$$P(\lambda) = (\lambda - e^{2t})(\lambda - e^{-t})^2 = \lambda^3 - \lambda^2(2e^{-t} + e^{2t}) + \lambda(e^{-2t} + 2e^{t}) - 1 \; . \tag{B.1.17}$$

The coefficients are given by sums and products of roots. We can use Lemma (2.2) in [97]. Let

$$P(\lambda) = \lambda^3 - k\lambda^2 + l\lambda - 1 \in \mathbb{Z}[\lambda] \, . \tag{B.1.18}$$

Then $P(\lambda)$ has a double root $\lambda_0 \in \mathbb{R}$ if and only if $\lambda_0 = +1$ or $\lambda_0 = -1$ for which $P(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda - 1$ or $P(\lambda) = \lambda^3 + \lambda^2 - \lambda - 1$ respectively.

In our case, we find the double root e^{-t}. This means the only way to have this polynomial with integer coefficients is to set $t = 0$. Then we can conclude there is no lattice.

Note that the same reasoning can be done for $\mathfrak{g}_{4.5}^{-\frac{1}{2},-\frac{1}{2}}$.

[6] A naïve reason one could think of would be that it is due to the off-diagonal piece, but as we are going to show, this piece actually does not contribute.

Algebras admitting a lattice

We present here a list of indecomposable solvable, non-nilpotent unimodular algebras[7] that admit a lattice (at least for certain values of the parameters p, q, r appearing, for instance the values chosen in table C.1.2.). For dimension up to 4 the algebras are almost nilpotent or almost abelian. For dimension 5 and 6, only almost abelian algebras have been considered. The list might not be complete only for the six-dimensional indecomposable algebras: for the other algebras in that case, we do not know if a lattice exists.

Name	Algebra	
$\mathfrak{g}_{3.4}^{-1}$	$[X_1, X_3] = X_1, [X_2, X_3] = -X_2$	alm. ab.
$\mathfrak{g}_{3.5}^{0}$	$[X_1, X_3] = -X_2, [X_2, X_3] = X_1$	alm. ab.
$\mathfrak{g}_{4.5}^{p,-p-1}$	$[X_1, X_4] = X_1, [X_2, X_4] = pX_2, [X_3, X_4] = -(p+1)X_3,\ -\frac{1}{2} \leq p < 0$	alm. ab.
$\mathfrak{g}_{4.6}^{-2p,p}$	$[X_1, X_4] = -2pX_1, [X_2, X_4] = pX_2 - X_3, [X_3, X_4] = X_2 + pX_3,\ p > 0$	alm. ab.
$\mathfrak{g}_{4.8}^{-1}$	$[X_2, X_3] = X_1, [X_2, X_4] = X_2, [X_3, X_4] = -X_3$	alm. nil.
$\mathfrak{g}_{4.9}^{0}$	$[X_2, X_3] = X_1, [X_2, X_4] = -X_3, [X_3, X_4] = X_2$	alm. nil.

Table B.1: Indecomposable non-nilpotent solvable unimodular algebras up to dimension 4, that admit a lattice

Name	Algebra
$\mathfrak{g}_{5.7}^{p,q,r}$	$[X_1, X_5] = X_1, [X_2, X_5] = pX_2, [X_3, X_5] = qX_3, [X_4, X_5] = rX_4,$ $-1 \leq r \leq q \leq p \leq 1\ ,\ pqr \neq 0\ ,\ p+q+r+1 = 0$
$\mathfrak{g}_{5.8}^{-1}$	$[X_2, X_5] = X_1, [X_3, X_5] = X_3, [X_4, X_5] = -X_4$
$\mathfrak{g}_{5.13}^{-1-2q,q,r}$	$[X_1, X_5] = X_1, [X_2, X_5] = -(1+2q)X_2, [X_3, X_5] = qX_3 - rX_4, [X_4, X_5] = rX_3 + qX_4,$ $-1 \leq q \leq 0\ ,\ q \neq -\frac{1}{2}\ ,\ r \neq 0$
$\mathfrak{g}_{5.14}^{0}$	$[X_2, X_5] = X_1, [X_3, X_5] = -X_4, [X_4, X_5] = X_3$
$\mathfrak{g}_{5.15}^{-1}$	$[X_1, X_5] = X_1, [X_2, X_5] = X_1 + X_2, [X_3, X_5] = -X_3, [X_4, X_5] = X_3 - X_4$
$\mathfrak{g}_{5.17}^{p,-p,r}$	$[X_1, X_5] = pX_1 - X_2, [X_2, X_5] = X_1 + pX_2, [X_3, X_5] = -pX_3 - rX_4, [X_4, X_5] = rX_3 - pX_4,$ $r \neq 0$
$\mathfrak{g}_{5.18}^{0}$	$[X_1, X_5] = -X_2, [X_2, X_5] = X_1, [X_3, X_5] = X_1 - X_4, [X_4, X_5] = X_2 + X_3$

Table B.2: Indecomposable solvable unimodular almost abelian algebras of dimension 5, that admit a lattice

Name	Algebra
$\mathfrak{g}_{6.3}^{0,-1}$	$[X_2, X_6] = X_1, [X_3, X_6] = X_2, [X_4, X_6] = X_4, [X_5, X_6] = -X_5$
$\mathfrak{g}_{6.10}^{0,0}$	$[X_2, X_6] = X_1, [X_3, X_6] = X_2, [X_4, X_6] = -X_5, [X_5, X_6] = X_4$

Table B.3: Indecomposable solvable unimodular almost abelian algebras of dimension 6, for which we know a lattice exists

B.2 The projection basis for intermediate $SU(2)$ structure solutions

In section 3.4.1, we explained that the good variables to use for intermediate $SU(2)$ structure solutions were those of the projection basis:

$$\mathrm{Re}(z),\ \mathrm{Im}(z),\ \mathrm{Im}(\omega),\ \mathrm{Re}(\omega)_{||},\ \mathrm{Re}(\omega)_{\perp},\ (j_{||},\ j_{\perp})\ ,$$

[7] For a list of six-dimensional nilpotent algebras, see for instance [29, 93].

where $j_{||}$, $j_\perp$ can eliminated using the projection conditions (3.4.12). Here, we rewrite the different equations to be solved in terms of these variables.

B.2.1 $SU(2)$ structure conditions

We rewrite the $SU(2)$ structure conditions implying the compatibility conditions (see appendix A.2.2), namely (A.2.2), (A.2.7), (A.2.8) and (A.2.9). To do so we also use the projection conditions (3.4.12). The $SU(2)$ structure conditions (A.2.2) and (A.2.7), for $O6/O5$ (upper/lower), are equivalent to

$$\begin{aligned}
&\mathrm{Im}(\omega) \wedge \mathrm{Re}(\omega)_{||} = 0 \ , \\
&\mathrm{Im}(\omega) \wedge \mathrm{Re}(\omega)_{\perp} = 0 \ , \\
&\mathrm{Re}(\omega)_{||} \wedge \mathrm{Re}(\omega)_{\perp} = 0 \ , \\
&\mathrm{Re}(\omega)_{||}^2 = \left(\frac{k_\perp}{k_{||}}\right)^{\mp 2} \mathrm{Re}(\omega)_{\perp}^2 \ , \\
&\mathrm{Re}(\omega)_{||}^2 + \mathrm{Re}(\omega)_{\perp}^2 = \mathrm{Im}(\omega)^2 \ .
\end{aligned} \tag{B.2.1}$$

We do not get any new condition from (A.2.8) and (A.2.9). This is because z, $\mathrm{Im}(\omega)$, $\mathrm{Re}(\omega)_{||}$, $\mathrm{Re}(\omega)_\perp$ define, modulo a rescaling, a new $SU(2)$ structure obtained by a rotation from the previous one. So it is natural [31] to have the five previous "wedge conditions", and only them.

We recall that this last set of conditions, together with the projection conditions, is then enough to get all the compatibility conditions except from (A.2.13). Using the last relations and the projection basis, we can also rewrite (A.2.13), for an O6/O5, as

$$\begin{aligned}
&\mathrm{Re}(z) \wedge \mathrm{Im}(z) \wedge \left(\mathrm{Im}(\omega)^2 + \frac{1}{k_{||}^2}\,\mathrm{Re}(\omega)_{||/\perp}^2\right) \neq 0 \ , \\
&\Leftrightarrow \mathrm{Re}(z) \wedge \mathrm{Im}(z) \wedge \mathrm{Re}(\omega)_{||}^2 \neq 0 \ .
\end{aligned} \tag{B.2.2}$$

B.2.2 SUSY conditions derivation

In section 2.4.2, we gave the SUSY equations reformulated in terms of the GCG pure spinors: (2.4.18), (2.4.19), and (2.4.20). Furthermore, we motivated the use of the projection basis variables, which are equivalent to the dielectric variables (3.4.22). The dielectric variables also provide an $SU(2)$ structure and verify the following $SU(2)$ structure conditions

$$\begin{aligned}
&j_D \wedge \omega_{Dr} = 0 \qquad j_D \wedge \omega_{Di} = 0 \qquad \omega_{Dr} \wedge \omega_{Di} = 0 \\
&j_D^2 = \omega_{Dr}^2 = \omega_{Di}^2 \neq 0 \\
&z \llcorner j_D = 0 \qquad z \llcorner \omega_{Dr} = 0 \qquad z \llcorner \omega_{Di} = 0
\end{aligned} \tag{B.2.3}$$

where we introduced to lighten notations

$$\mathrm{Re}(\omega_D) = \omega_{Dr} \ , \qquad \mathrm{Im}(\omega_D) = \omega_{Di} \ , \qquad \mathrm{Re}(z) = z_r \ , \qquad \mathrm{Im}(z_D) = z_i \ . \tag{B.2.4}$$

Here we will use the pure spinors given in terms of the dielectric variables by

$$\begin{aligned}
\Phi_+ &= \frac{|a|^2}{8} e^{i\theta_+} e^{\frac{1}{2} z \wedge \overline{z}} (k_{||} - i j_D + k_\perp \omega_{Di} - \frac{1}{2} k_{||} j_D^2) \\
\Phi_- &= -\frac{|a|^2}{8} z \wedge (k_\perp + i\omega_{Dr} - k_{||}\omega_{Di} - \frac{1}{2} k_\perp j_D^2) \ ,
\end{aligned} \tag{B.2.5}$$

where we took $\theta_- = 0$. Plugging these pure spinors in (2.4.18), (2.4.19), and (2.4.20), we can derive the equations to solve. Let us first consider coefficients to be a priori non-constant. We only assume that e^A, a, and $k_{||}$ are not vanishing. The last condition forbids to obtain orthogonal $SU(2)$ structures (more precisely it forbids to solve our equations at points where we have orthogonal $SU(2)$ structures). Due to these assumptions, it will not be possible to get diverging-dilaton-like solutions (see (B.2.8) and (B.2.13)). Note as well the following useful relation:

$$\mathrm{d}(k_{||}) = -\frac{k_\perp}{k_{||}} d(k_\perp) \ . \tag{B.2.6}$$

IIA equations

In IIA, we first get with (2.4.18) the following equation:

$$\mathrm{d}(|a|^2 e^{i\theta_+} e^{2A-\phi} k_{||}) = 0 \ . \tag{B.2.7}$$

Decomposing on real and imaginary part, given the assumptions made, we first get that θ_+ should be constant. This could be surprising because of the Frey-Graña [98] solutions, which do have such a phase. But these solutions have an $SU(3)$ structure in IIB, i.e. the typical T-dual situation of an orthogonal $SU(2)$ structure in IIA. As we assumed $k_{||} \neq 0$, there is no contradiction. So θ_+ is taken to be constant. We solve this equation by introducing $\tilde{g}_s$ as an integration constant and get

$$e^\phi = \tilde{g}_s k_{||} e^{2A} |a|^2 \ . \tag{B.2.8}$$

We plug this result into the other equations derived from (2.4.18) and (2.4.19) and get these SUSY equations:

$$\begin{aligned}
&\mathrm{d}\left(e^{-A}\frac{k_\perp}{k_{||}} z_r\right) = 0 \\
&\mathrm{d}\left(\frac{1}{k_{||}} j_D + z_r \wedge z_i\right) = 0 \\
&\mathrm{d}\left(\frac{k_\perp}{k_{||}} \omega_{Di}\right) = H \\
&\mathrm{d}\left(\frac{e^{-A}}{k_{||}} (z_r \wedge \omega_{Di} k_{||} + z_i \wedge \omega_{Dr})\right) = e^{-A}\frac{k_\perp}{k_{||}} z_r \wedge H \\
&\mathrm{d}\left(\frac{k_\perp}{k_{||}} z_r \wedge z_i \wedge \omega_{Di}\right) = \frac{1}{k_{||}} H \wedge (j_D + k_{||} z_r \wedge z_i) \\
&\frac{1}{2}\mathrm{d}(j_D^2) + \mathrm{d}\left(\frac{1}{k_{||}} z_r \wedge z_i \wedge j_D\right) = -\frac{k_\perp}{k_{||}} H \wedge \omega_{Di} \\
&-\frac{e^{-A}}{k_{||}}\frac{k_\perp}{2} z_r \wedge \mathrm{d}(j_D^2) = \frac{e^{-A}}{k_{||}} H \wedge (z_r \wedge \omega_{Di} k_{||} + z_i \wedge \omega_{Dr}) \ .
\end{aligned} \tag{B.2.9}$$

The equation (2.4.20) gives the definitions of the RR fluxes:

$$\begin{aligned}
&F_6 = 0 \\
&\mathrm{d}\left(e^{A}\frac{k_\perp}{k_{||}} z_i\right) = -\tilde{g}_s |a|^2 e^{3A} * F_4 \\
&\mathrm{d}\left(\frac{e^{A}}{k_{||}} (-z_i \wedge \omega_{Di} k_{||} + z_r \wedge \omega_{Dr})\right) + e^{A}\frac{k_\perp}{k_{||}} z_i \wedge H = \tilde{g}_s |a|^2 e^{3A} * F_2 \\
&\mathrm{d}\left(\frac{e^{A}}{k_{||}}\frac{k_\perp}{2} z_i \wedge j_D^2\right) + \frac{e^{A}}{k_{||}} H \wedge (-z_i \wedge \omega_{Di} k_{||} + z_r \wedge \omega_{Dr}) = \tilde{g}_s |a|^2 e^{3A} * F_0 \ .
\end{aligned} \tag{B.2.10}$$

Using the (wedge) structure conditions (B.2.3), the equations (B.2.9) get simplified. Furthermore, using the lower order form SUSY equations, the higher orders (5- and 6-forms equations) are automatically satisfied. So the set of equations gets reduced to

$$\begin{aligned}
&\mathrm{d}\left(\frac{k_\perp}{k_{||}}\omega_{Di}\right) = H \\
&\mathrm{d}\left(\frac{k_\perp}{k_{||}}e^{-A}z_r\right) = 0 \\
&\mathrm{d}\left(\frac{1}{k_{||}}j_D + z_r\wedge z_i\right) = 0 \\
&\mathrm{d}\left(\frac{e^{-A}}{k_{||}^2}(z_r\wedge\omega_{Di} + k_{||}z_i\wedge\omega_r)\right) = 0\ .
\end{aligned} \tag{B.2.11}$$

IIB equations

We use the same procedure as in IIA. (2.4.19) first gives:

$$\mathrm{d}(|a|^2\cos(\theta_+)e^{A-\phi}k_{||}) = 0\ . \tag{B.2.12}$$

Here θ_+ can vary as there is only a real part in this equation. To go further, we have to restrict a possible resolution to the intervals where $\cos(\theta_+) \neq 0$, forbidding to get so-called type C solutions on these intervals. We solve this equation by introducing $\tilde{g}_s$ as an integration constant and get

$$e^\phi = \tilde{g}_s k_{||}e^A|a|^2\cos(\theta_+)\ . \tag{B.2.13}$$

We simplify notations with $\cos(\theta_+) = c_\theta$, $\tan(\theta_+) = t_\theta$. We plug this into the other equations derived from (2.4.18) and (2.4.19) and get

$$\begin{aligned}
&\mathrm{d}\left(e^A\frac{k_\perp}{k_{||}c_\theta}z_r\right) = 0\ , \qquad \mathrm{d}\left(e^A\frac{k_\perp}{k_{||}c_\theta}z_i\right) = 0 \\
&\mathrm{d}\left(\frac{k_\perp}{k_{||}}\omega_{Di} + \frac{t_\theta}{k_{||}}(k_{||}z_r\wedge z_i + j_D)\right) = H \\
&\mathrm{d}\left(\frac{e^A}{k_{||}c_\theta}(z_r\wedge\omega_{Di}k_{||} + z_i\wedge\omega_{Dr})\right) = e^A\frac{k_\perp}{k_{||}c_\theta}z_r\wedge H \\
&\mathrm{d}\left(\frac{e^A}{k_{||}c_\theta}(-z_i\wedge\omega_{Di}k_{||} + z_r\wedge\omega_{Dr})\right) = -e^A\frac{k_\perp}{k_{||}c_\theta}z_i\wedge H \\
&\mathrm{d}\left(\frac{1}{2}j_D^2 + \frac{1}{k_{||}}z_r\wedge z_i\wedge j_D - t_\theta\frac{k_\perp}{k_{||}}z_r\wedge z_i\wedge\omega_{Di}\right) = -\frac{1}{k_{||}}H\wedge(k_\perp\omega_{Di} + t_\theta(k_{||}z_r\wedge z_i + j_D)) \\
&\frac{e^A}{k_{||}c_\theta}\frac{k_\perp}{2}z_i\wedge\mathrm{d}(j_D^2) = \frac{e^A}{k_{||}c_\theta}H\wedge(-z_i\wedge\omega_{Di}k_{||} + z_r\wedge\omega_{Dr}) \\
&-\frac{e^A}{k_{||}c_\theta}\frac{k_\perp}{2}z_r\wedge\mathrm{d}(j_D^2) = \frac{e^A}{k_{||}c_\theta}H\wedge(z_r\wedge\omega_{Di}k_{||} + z_i\wedge\omega_{Dr})\ .
\end{aligned} \tag{B.2.14}$$

The equation (2.4.20) gives the definitions of the RR fluxes:

$$
\begin{aligned}
&\mathrm{d}(e^{2A}t_\theta) = \tilde{g}_s|a|^2 e^{3A} * F_5 \\
&\mathrm{d}\left(\frac{e^{2A}}{k_{||}}(z_r \wedge z_i k_{||} + j_D - t_\theta k_\perp \omega_{Di})\right) + e^{2A} t_\theta H = \tilde{g}_s|a|^2 e^{3A} * F_3 \\
&\mathrm{d}\left(\frac{e^{2A}}{k_{||}}\left(-z_r \wedge z_i \wedge \omega_{Di} k_\perp - t_\theta (z_r \wedge z_i \wedge j_D + \frac{k_{||}}{2} j_D^2)\right)\right) + \frac{e^{2A}}{k_{||}} H \wedge (z_r \wedge z_i k_{||} + j_D - t_\theta k_\perp \omega_{Di}) \\
&\qquad = \tilde{g}_s|a|^2 e^{3A} * F_1
\end{aligned}
\tag{B.2.15}
$$

As in IIA, the set of equations gets simplified (in particular lower order form equations of SUSY imply the 6-form equations) to

$$
\begin{aligned}
&\mathrm{d}\left(\frac{k_\perp}{k_{||}}\omega_{Di} + t_\theta\left(\frac{1}{k_{||}} j_D + z_r \wedge z_i\right)\right) = H \\
&\mathrm{d}\left(\frac{k_\perp e^A}{k_{||} c_\theta} z_r\right) = 0 \ , \qquad \mathrm{d}\left(\frac{k_\perp e^A}{k_{||} c_\theta} z_i\right) = 0 \\
&\mathrm{d}\left(\frac{e^A}{k_{||} c_\theta}\left(z_r \wedge \left(\frac{1}{k_{||}}\omega_{Di} + t_\theta \frac{k_\perp}{k_{||}} j_D\right) + z_i \wedge \omega_r\right)\right) = 0 \\
&\mathrm{d}\left(\frac{e^A}{k_{||} c_\theta}\left(z_i \wedge \left(\frac{1}{k_{||}}\omega_{Di} + t_\theta \frac{k_\perp}{k_{||}} j_D\right) - z_r \wedge \omega_r\right)\right) = 0 \\
&\mathrm{d}\left(\frac{1}{c_\theta^2 k_{||}} j_D \wedge \left(\frac{1}{2k_{||}} j_D + z_r \wedge z_i\right)\right) = 0 \ .
\end{aligned}
\tag{B.2.16}
$$

Towards intermediate $SU(2)$ structure solutions

Looking for intermediate $SU(2)$ structure solutions, one can further simplify the previous set of equations. First we take $k_{||}$ and $k_\perp$ to constant and non-zero. We choose θ_+ to be constant (in IIB). Furthermore, we will choose $|a|^2 = e^A$, and go to the large volume limit, i.e. $A = 0$. Indeed in the main part, solutions are found in this regime. All these restrictions lead to a constant dilaton. Therefore, in both theories, we can reabsorb all the constant factors into one: $e^\phi = g_s$.

We do not really need to fix the remaining freedom in θ_+, except that in IIB, we can use the O5 condition on this phase: $e^{i\theta_+} = \pm 1$. Then, coming back to the projection basis variables via (3.4.22), we obtain the SUSY equations of the main part (3.4.24) and (3.4.25).

B.3 Calibrated supersymmetric sources

In this appendix, we briefly recall some results about calibrations of supersymmetric sources [45, 30, 46, 11], and motivate the normalisation condition (3.3.6). For simplicity, we will not consider any flux pulled-back to the source, nor any worldvolume gauge field. Consider a supersymmetric source wrapping a k-dimensional subspace Σ of a d-dimensional internal space M. Then, the volume form of Σ on the source worldvolume is given by the pullback of one of the GCG pure spinors

$$ i^*[\mathrm{Im}(\Phi_2)]|_k = \frac{|a|^2}{8}\sqrt{|i^*[g]|}\ d^\Sigma \xi \ , \tag{B.3.1} $$

where i is the embedding of Σ into M, g the internal metric, ξ the coordinates on Σ, and the index k indicates the restriction to k-forms (see also (5.1.3)). In particular, if one has a trivial embedding, then

$$ \mathrm{Im}(\Phi_2)|_k = \frac{|a|^2}{8} V_\Sigma \ , \tag{B.3.2} $$

where V_Σ is the volume form of Σ. More generally, one can introduce a current j_Σ, defined in our conventions as (the Mukai pairing was defined in (2.4.14))

$$\int_M \langle j_\Sigma, f\rangle = \int_\Sigma i^*[f] \ , \tag{B.3.3}$$

for a polyform f of M. The current acts as a dimensionless δ, or equivalently is given by a Poincaré dual with respect to the Mukai pairing. For $i^*[\mathrm{Im}(\Phi_2)]|_{\Sigma_l}$ for a source l, the current j_{Σ_l} turns out to be related to the right handside of the BI, i.e. it localizes the sources. In the smeared approximation (the δ functions are taken to 1), j_{Σ_l} is then proportional to the covolume V^l of the source l (see (3.3.5)), which should also be given by the Poincaré dual of the source volume form.

More precisely, in our conventions, the Poincaré dual with respect to the Mukai pairing can be read in the following identity

$$\langle *\lambda(V_{\Sigma_l}), V_{\Sigma_l}\rangle \ = V \ , \tag{B.3.4}$$

where V is the internal space volume form. So we choose the covolumes in (3.3.5) to satisfy

$$V^l = *\lambda(V_{\Sigma_l}) \ , \tag{B.3.5}$$

and we deduce, in the large volume limit

$$\left\langle V^l, e^{-\phi}\mathrm{Im}(\Phi_2)\right\rangle = \frac{1}{8g_s}\ V \ , \tag{B.3.6}$$

where we introduced the dilaton for further use.

Note this normalisation may be refined, to take into account some possible forgotten volume factors. But all these factors are positive, so they are not changing the sign of the charges, which is what matters in the end. Furthermore, one obtains

$$\int_{M_6} \left\langle V^l, e^{3A-\phi}\mathrm{Im}(\Phi_2)\right\rangle \ > 0 \ . \tag{B.3.7}$$

Using this condition and our conventions for the Hodge star, it can be shown as in [29] that $\sum_l Q_l < 0$, and so recover the need for orientifolds as sources, because of their negative charge.

Appendix C

Twist transformation

In this appendix, we come back to the twist transformation discussed in chapter 4 and to the solutions on solvmanifolds mentioned there. We first give a more detailed twist construction of the one-forms of an (almost) nilpotent solvable group. Then we give a list of solvmanifolds in terms of their globally defined one-forms. This allows one to look for solutions on these manifolds. Then we study the possible non-geometric T-duals of the solutions found on solvmanifolds. Finally, we discuss an extension of the twist transformation in heterotic string to the gauge bundle, by an extended action $O(d+16, d+16)$. This extension allows to transform the gauge fields directly.

C.1 Construction of one-forms and basis for algebras

C.1.1 Algorithmic construction of the one-forms of a solvable group

Let us consider a connected and simply-connected six-dimensional solvable group G (see section 3.2 and appendix B.1 for definitions and properties). As a manifold, its tangent bundle at the identity is given by $T_eG \approx \mathfrak{g}$, and has a basis of vectors E_a ($a = 1 \dots 6$) satisfying

$$[E_b, E_c] = f^a{}_{bc} E_a \ . \tag{C.1.1}$$

We will focus on the dual basis of one-forms e^a on the cotangent bundle $g^* \approx T_eG^*$, which verify the Maurer-Cartan equation

$$\mathrm{d}e^a = -\frac{1}{2} f^a{}_{bc} e^b \wedge e^c = -\sum_{b<c} f^a{}_{bc}\ e^b \wedge e^c \ . \tag{C.1.2}$$

We want to consider a transformation A relating the one-forms of $\mathbb{R}^6$ to those of G:

$$A \begin{pmatrix} \mathrm{d}x^1 \\ \vdots \\ \mathrm{d}x^6 \end{pmatrix} = \begin{pmatrix} e^1 \\ \vdots \\ e^6 \end{pmatrix} \ . \tag{C.1.3}$$

Clearly the one-forms in (C.1.3) must satisfy the corresponding[1] Maurer-Cartan equation.

The matrix A should reproduce the different fibrations of the solvable group (the bundle structure is manifest in the Maurer-Cartan equations). Given the general form of solvable groups (a nilradical subgroup N and an abelian left over subgroup $G/N = \mathbb{R}^{\dim\ G - \dim\ N}$, see section 3.2), we will consider A to be a product of two pieces:

$$A = \left(\begin{array}{c|c} A_N & 0 \\ \hline 0 & \mathbb{I}_{6-\dim\ N} \end{array} \right) \left(\begin{array}{c|c} A_M & 0 \\ \hline 0 & \mathbb{I}_{6-\dim\ N} \end{array} \right) \ , \tag{C.1.4}$$

[1] Whether the exterior derivative is defined on these new forms will not be treated (see footnote 2 in section 3.2): we will just define it as the exterior derivative of $\mathbb{R}^6$ acting on the left handside of (C.1.3).

where we take A_M and A_N to be $\dim N \times \dim N$ matrices, and we put the abelian directions of $\mathbb{R}^{\dim\ G-\dim\ N}$ in the last entries. A_M will provide the non-trivial fibration of N over $\mathbb{R}^{\dim\ G-\dim\ N}$, the Mostow bundle fibration of the solvmanifold for the compact case, as explained in section 3.2. In turn, A_N will provide fibrations inside N, the fibrations within the nilmanifold piece for the compact case. If the solvable group is nilpotent, then we take A_M to be the identity.

To explicitly construct the matrices A_M and A_N we will now restrict ourselves to $G = N$ (nilpotent) or $G = \mathbb{R} \ltimes_\mu N$ (almost nilpotent).

Mostow bundle structure: A_M

We focus on the case of an almost nilpotent group. We identify the $\mathbb{R}$ subalgebra with the direction x^6. Then we take $\partial_t = \partial_6$ the basis for the $\mathbb{R}$ subalgebra, and the corresponding one-form $\mathrm{d}x^6 = \mathrm{d}t$. Then we define

$$A_M = Ad_{e^{-t\partial_t}}(\mathfrak{n}) = e^{-t\ ad_{\partial_t}(\mathfrak{n})}\ , \tag{C.1.5}$$

and

$$e^i = (A_M)^i{}_k\, \mathrm{d}x^k\ . \tag{C.1.6}$$

Let us prove that this action will give forms which do verify the Maurer-Cartan equation. Consider first the simpler case of an almost abelian group, i.e. with $N = \mathbb{R}^5$, which has $A_N = \mathbb{I}_N$. Then

$$\begin{aligned}
\mathrm{d}e^i &= \mathrm{d}(e^{-t\ ad_{\partial_t}})^i{}_k \wedge \mathrm{d}x^k \\
&= -\mathrm{d}t \wedge (ad_{\partial_t} e^{-t\ ad_{\partial_t}})^i{}_k \mathrm{d}x^k \\
&= -\mathrm{d}t \wedge (ad_{\partial_t})^i{}_j (e^{-t\ ad_{\partial_t}})^j{}_k \mathrm{d}x^k \\
&= -\mathrm{d}t \wedge (ad_{\partial_t})^i{}_j e^j \\
\mathrm{d}e^i &= -f^i{}_{tj}\ \mathrm{d}t \wedge e^j\ .
\end{aligned} \tag{C.1.7}$$

The fact that we used the adjoint action allows to easily verify the Maurer-Cartan equations.

Expression (C.1.5) for the matrix A_M holds also for the more general case of almost nilpotent algebras. In this case the Maurer-Cartan equations have component in direction $\mathrm{d}t$ and also in the directions of the nilradical. The t dependence is always determined by A_M and hence it is not modified by the presence of a non-trivial nilradical. The form of the nilradical matrix, A_N, is given below.

Nilmanifold fibration structure: A_N

The matrix A_N should reproduce the iterated fibration structure of N. The iterated structure is related to the descending serie of $\mathfrak{n}$ (see appendix B.1) noted:

$$\mathfrak{n}^{k=0\dots p} \text{ with } \mathfrak{n}^0 = \mathfrak{n}\ ,\ \mathfrak{n}^p = \{0\}\ .$$

Every $\mathfrak{n}^k$ is an ideal of $\mathfrak{g}$, so $\forall k \geq 1\ ,\ \mathfrak{n}^k = [\mathfrak{n}, \mathfrak{n}^{k-1}] \subset [\mathfrak{g}, \mathfrak{n}^{k-1}] \subset \mathfrak{n}^{k-1}$. Let us now define another serie:

$$\text{For } 1 \leq k \leq p,\ s^k = \{E \in \mathfrak{n}^{k-1} \text{ with } E \notin \mathfrak{n}^k\}\ . \tag{C.1.8}$$

Let us prove some property of this serie. Assume that $\exists X \in s^p \bigcap s^q\ ,\ p > q$ with $X \neq 0$. Then $X \in \mathfrak{n}^{p-1} \subset \mathfrak{n}^{p-2} \subset \cdots \subset \mathfrak{n}^q \subset \mathfrak{n}^{q-1}$. So $X \in \mathfrak{n}^{q-1}$ and $X \in \mathfrak{n}^q$, so $X \notin s^q$, which is a contradiction. So $s^p \bigcap s^q = \{0\}$ for $p \neq q$. Furthermore, we always have $s^p = \mathfrak{n}^{p-1}$. So $s^{p-1} \bigcup s^p = s^{p-1} \bigcup \mathfrak{n}^{p-1} = \mathfrak{n}^{p-2} \bigcup \mathfrak{n}^{p-1} = \mathfrak{n}^{p-2}$. Assume that $s^k \bigcup s^{k+1} \bigcup \cdots \bigcup s^{p-1} \bigcup s^p = \mathfrak{n}^{k-1}$. Then $s^{k-1} \bigcup s^k \bigcup \cdots \bigcup s^{p-1} \bigcup s^p = s^{k-1} \bigcup \mathfrak{n}^{k-1} = \mathfrak{n}^{k-2} \bigcup \mathfrak{n}^{k-1} = \mathfrak{n}^{k-2}$. So by recurrence, we get that $\bigcup_{k=1\dots p} s^k = \mathfrak{n}$. In other words, each element of $\mathfrak{n}$ appears in one and only one element of the serie $s^{\{k\}}$.

Let us give an example: consider the five-dimensional solvable algebra $(0, 31, -21, 23, 24)$ (notations of section 3.2). We have

$$\begin{array}{rl} \mathfrak{g} = \{1,2,3,4,5\} \quad , & \mathfrak{n} = \{2,3,4,5\} \ , \ \mathfrak{n}^1 = \{4,5\} \ , \ \mathfrak{n}^2 = \{5\} \ , \ \mathfrak{n}^3 = \{0\} \\ & s^1 = \{2,3\} \ , \ s^2 = \{4\} \ , \ s^3 = \{5\} \ . \end{array}$$

The descending serie of $\mathfrak{n}$ is known to be related to the fibration structure of the nilpotent group: each element gives a further fibration. Now we understand that the serie $s^{\{k\}}$ gives us what directions are fibered at each step. The correspondence between basis, fibers and series for a general iteration is given in the following diagram (of course it should be understood in terms of group elements instead of algebra elements as given here, see [38]):

$$\begin{array}{ccc} \mathcal{F}^{p-1} = s^p & \hookrightarrow & \mathcal{M}^{p-1} = \mathfrak{n} \\ & & \downarrow \\ \mathcal{F}^{p-2} = s^{p-1} & \hookrightarrow & \mathcal{M}^{p-2} = \mathcal{B}^{p-1} \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ \mathcal{F}^{2} = s^{3} & \hookrightarrow & \mathcal{M}^{2} = \mathcal{B}^{3} \\ & & \downarrow \\ \mathcal{F}^{1} = s^{2} & \hookrightarrow & \mathcal{M}^{1} = \mathcal{B}^{2} \\ & & \downarrow \\ & & \mathcal{B}^{1} = s^{1} \end{array}$$

We see the unique decomposition of $\mathfrak{n}$ into the serie $s^{\{k\}}$. We have $\mathcal{B}^i = \bigcup_{k=1\dots i} s^k$ and $\mathcal{F}^i = s^{i+1}$.

In the general case of an iteration, we consider a product of several operators, each of them giving one fibration of the iteration:

$$A_N = A_{p-1} \dots A_1 \ , \ A_i = e^{-\frac{1}{2} f_i} \qquad (\text{for } p = 1,\ \mathfrak{n} = \mathbb{R}^5 \text{ and } A_N = 1) \ ,$$

with $f_i \in \mathrm{End}(\mathfrak{n})$:

$$\begin{array}{rcl} \text{For } i = 1 \dots p-1 \ , \ f_i : \mathfrak{n} & \longrightarrow & \mathfrak{n} \\ X & \longmapsto & Y = ad_{\mathcal{B}^i}(X) \text{ if } X \in \mathcal{B}^i \text{ and } ad_{\mathcal{B}^i}(X) \in \mathcal{F}^i \ , \\ & & Y = 0 \text{ otherwise} \ . \end{array} \tag{C.1.9}$$

We choose to give a basis of $\mathfrak{n}$ in the order given by $s^1, s^2, \dots, s^p$, and in each s^k we can choose some order for the elements. Then in that basis, f_i, as a matrix, is an off-diagonal block with lines corresponding to $\mathcal{F}^i = s^{i+1}$ and columns to $\mathcal{B}^i = \bigcup_{k=1\dots i} s^k$. Then A_i is the same plus the identity. Furthermore, the block depends on parameters a^j of a generic element $a^j E_j$ of $\mathcal{B}^i$, and we have $ad_{a^j E_j \in \mathcal{B}^i} = a^j ad_{E_j \in \mathcal{B}^i}$. So for instance for the previous algebra, we get:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2}a^3 & -\frac{1}{2}a^2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2}a^4 & 0 & -\frac{1}{2}a^2 & 1 \end{pmatrix} \ . \tag{C.1.10}$$

The parameters a^j can be understood as a coordinate along E_j, so they are such that $\mathrm{d}a^j = e^j$, dual of E_j.

Let us prove that the operator A_r gives the fibration of directions of $\mathcal{F}^r$ over a base $\mathcal{B}^r$, and the correct corresponding Maurer-Cartan equation. As explained, an element of A_r is given by:

$$(A_r)^i{}_k = \delta^i_k - \frac{1}{2}\sum_{j\in\mathcal{B}^r} a^j (ad_{E_j})^i{}_k\ \Theta(i\in\mathcal{F}^r)\Theta(k\in\mathcal{B}^r) = \delta^i_k - \frac{1}{2}\sum_{j,k\in\mathcal{B}^r} a^j f^i{}_{jk}\ \Theta(i\in\mathcal{F}^r)\ . \quad \text{(C.1.11)}$$

The forms on which we act with A_r at the step r of the iteration are labelled e^k, and they become after the operation $\tilde{e}^i$:

$$\tilde{e}^i = (A_r)^i{}_k\ e^k\ . \quad \text{(C.1.12)}$$

The directions we fiber with A_r are initially not fibered, so $e^{k\in\mathcal{F}^r} = \mathrm{d}x^k$. All the other directions are not modified by A_r, so in particular $\tilde{e}^{i\in\mathcal{B}^r} = e^{i\in\mathcal{B}^r}$. So the Maurer-Cartan equations of the forms not in $\mathcal{F}^r$ are not modified at this step. Their equation is then only modified at the step when they are fibered, so we do not have to consider it here. For the directions $\mathcal{F}^r$, we get:

$$\tilde{e}^{i\in\mathcal{F}^r} = e^{i\in\mathcal{F}^r} - \frac{1}{2}\sum_{j,k\in\mathcal{B}^r} a^j f^i{}_{jk}\ e^k = \mathrm{d}x^i - \frac{1}{2}\sum_{j,k} a^j f^i{}_{jk}\ e^k\ ,$$

where we dropped the restriction $j,k\in\mathcal{B}^r$ because due to the iterated structure, for $i\in\mathcal{F}^r$, $f^i{}_{jk}=0$ if k or $j\notin\mathcal{B}^r$. This operation then gives the fibration structure, since we can read the connection. We can verify that we have the correct Maurer-Cartan equation:

$$\mathrm{d}\tilde{e}^{i\in\mathcal{F}^r} = -\frac{1}{2}f^i{}_{jk}\ \mathrm{d}a^j\wedge e^k = -\frac{1}{2}f^i{}_{jk}\ e^j\wedge e^k = -\frac{1}{2}f^i{}_{jk}\ \tilde{e}^j\wedge\tilde{e}^k\ .$$

C.1.2 Six-dimensional solvmanifolds in terms of globally defined one-forms

In the following table we present all the solvmanifolds that we are able to construct. They have the form $G/\Gamma = H_1/\Gamma_1\times H_2/\Gamma_2$, i.e. they are products of (at most) two solvmanifolds. Each of these two solvmanifolds are constructed from the algebras given in the previous Tables (see appendix B.1.2) and the three-dimensional nilpotent algebra $\mathfrak{g}_{3.1}:(-23,0,0)$. In particular, these are indecomposable solvable algebras for which the group admits a lattice. The difference with respect to the tables of section B.1.2 is that the algebras are given here in terms of a basis of globally defined forms (see discussion in section 4.2). They are related by isomorphisms to the algebras given in the Tables of B.1.2. The fact the forms are globally defined is important for studying the compatibility of orientifold planes with the manifold and for finding solutions. For $\mathfrak{g}_{4.5}^{p,-p-1}\oplus\mathbb{R}^2$ and $\mathfrak{g}_{4.6}^{-2p,p}\oplus\mathbb{R}^2$, we were not able to find such a basis, even if a priori we expect it to exist.

The column Name indicates the label of the algebra and the corresponding solvmanifold. The column Algebra gives the corresponding six-dimensional algebra in terms of exterior derivative acting on the dual basis of globally defined one-forms (see section 3.2). The next two columns give the O5 and O6 planes that are compatible with the manifold. The column *Sp* indicates by a ✓ when the manifold is symplectic, according to [37, 40]. Note that the latter can be obtained as conditions for the pure spinors to solve the supersymmetry equations. In particular, for the even $SU(3)$ pure spinor $\Phi_+ = \frac{1}{8}e^{-iJ}$ the condition (see (4.4.4))

$$\mathrm{d}(O)\Phi_+ = 0 \quad \text{(C.1.13)}$$

is equivalent to the requirement that the manifold is symplectic, with O given in (4.3.25).

There is an additional subtlety for not completely solvable manifolds, when one looks for solutions on them. This is due to the lack of isomorphism between the cohomology groups $H^*(\mathfrak{g})$ and $H^*_{dR}(G/\Gamma)$ for not completely solvable manifolds (see footnote 2 in section 3.2). In other words, the Betti numbers for the Lie algebra cohomology give only the lower bound for the corresponding numbers for de Rham cohomology. When looking for e.g. symplectic manifolds, we have considered only the forms in $H^2(\mathfrak{g})$, and hence might have missed some candidate two-forms in $H^2_{dR}(G/\Gamma)$.

Name	Algebra	O5	O6	Sp
$\mathfrak{g}_{3.4}^{-1} \oplus \mathbb{R}^3$	$(q_1 23, q_2 13, 0, 0, 0, 0) \quad q_1, q_2 > 0$	14, 15, 16, 24, 25, 26, 34, 35, 36	123, 145, 146, 156, 245, 246, 256, 345, 346, 356	✓
$\mathfrak{g}_{3.5}^{0} \oplus \mathbb{R}^3$	$(-23, 13, 0, 0, 0, 0)$	14, 15, 16, 24, 25, 26, 34, 35, 36	123, 145, 146, 156, 245, 246, 256, 345, 346, 356	✓
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_{3.4}^{-1}$	$(-23, 0, 0, q_1 56, q_2 46, 0) \quad q_1, q_2 > 0$	14, 15, 16, 24, 25, 26, 34, 35, 36	-	✓
$\mathfrak{g}_{3.1} \oplus \mathfrak{g}_{3.5}^{0}$	$(-23, 0, 0, -56, 46, 0)$	14, 15, 16, 24, 25, 26, 34, 35, 36	-	✓
$\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_{3.5}^{0}$	$(q_1 23, q_2 13, 0, -56, 46, 0) \quad q_1, q_2 > 0$	14, 15, 16, 24, 25, 26, 34, 35, 36	-	✓
$\mathfrak{g}_{3.4}^{-1} \oplus \mathfrak{g}_{3.4}^{-1}$	$(q_1 23, q_2 13, 0, q_3 56, q_4 46, 0) \quad q_1, q_2, q_3, q_4 > 0$	14, 15, 16, 24, 25, 26, 34, 35, 36	-	✓
$\mathfrak{g}_{3.5}^{0} \oplus \mathfrak{g}_{3.5}^{0}$	$(-23, 13, 0, -56, 46, 0)$	14, 15, 16, 24, 25, 26, 34, 35, 36	-	✓
$\mathfrak{g}_{4.5}^{p,-p-1} \oplus \mathbb{R}^2$	?			-
$\mathfrak{g}_{4.6}^{-2p,p} \oplus \mathbb{R}^2$	?			-
$\mathfrak{g}_{4.8}^{-1} \oplus \mathbb{R}^2$	$(-23, q_1 34, q_2 24, 0, 0, 0) \quad q_1, q_2 > 0$	14, 25, 26, 35, 36	145, 146, 256, 356	-
$\mathfrak{g}_{4.9}^{0} \oplus \mathbb{R}^2$	$(-23, -34, 24, 0, 0, 0)$	14, 25, 26, 35, 36	145, 146, 256, 356	-
$\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}$	$(q_1 25, q_2 15, q_2 45, q_1 35, 0, 0) \quad q_1, q_2 > 0$	13, 14, 23, 24, 56	125, 136, 146, 236, 246, 345	✓
$\mathfrak{g}_{5.8}^{-1} \oplus \mathbb{R}$	$(25, 0, q_1 45, q_2 35, 0, 0) \quad q_1, q_2 > 0$	13, 14, 23, 24, 56	125, 136, 146, 236, 246, 345	✓
$\mathfrak{g}_{5.13}^{-1,0,r} \oplus \mathbb{R}$	$(q_1 25, q_2 15, -q_2 r45, q_1 r35, 0, 0)\ r \neq 0,\ q_1, q_2 > 0$	13, 14, 23, 24, 56	125, 136, 146, 236, 246, 345	✓
$\mathfrak{g}_{5.14}^{0} \oplus \mathbb{R}$	$(-25, 0, -45, 35, 0, 0)$	13, 14, 23, 24, 56	125, 136, 146, 236, 246, 345	✓
$\mathfrak{g}_{5.15}^{-1} \oplus \mathbb{R}$	$(q_1(25-35), q_2(15-45), q_2 45, q_1 35, 0, 0) \quad q_1, q_2 > 0$	14, 23, 56	146, 236	✓
$\mathfrak{g}_{5.17}^{p,-p,r} \oplus \mathbb{R}$	$(q_1(p25+35), q_2(p15+45), q_2(p45-15), q_1(p35-25), 0, 0)$ $r^2 = 1,\ q_1, q_2 > 0$	14, 23, 56 $p = 0$: 12, 34	146, 236 $p = 0$: 126, 135, 245, 346	✓
$\mathfrak{g}_{5.18}^{0} \oplus \mathbb{R}$	$(-25-35, 15-45, -45, 35, 0, 0)$	14, 23, 56	146, 236	✓
$\mathfrak{g}_{6.3}^{0,-1}$	$(-26, -36, 0, q_1 56, q_2 46, 0) \quad q_1, q_2 > 0$	24, 25	134, 135, 456	✓
$\mathfrak{g}_{6.10}^{0,0}$	$(-26, -36, 0, -56, 46, 0)$	24, 25	134, 135, 456	✓

C.2 T-dualising solvmanifolds

T-duality has been extensively used in flux compactifications in order to obtain solutions on nilmanifolds. Being iterations of torus bundles, these are obtainable from torus solutions with an appropriate B-field (the contraction of H with the isometry vectors should be a closed horizontal two-form that can be thought as a curvature of the dual torus bundle.). Correspondingly, the structure constants ${f^a}_{bc}$ have also a T-duality friendly form. For any upper index there is a well-defined isometry vector ∂_a with respect to which one can perform an (unobstructed) T-duality.

In this appendix we would like to study some aspects of T-duality for solvmanifolds (see section 3.2 and appendix B.1 for definitions and properties). In this case, the situation is more complicated. For instance, it can happen that the structure constants have the same index in the upper and lower position ${f^a}_{ac}$ and are not fully antisymmetric. Put differently, most of our knowledge about the global aspects of T-duality comes from the study of its action on (iterations of) principal $U(1)$ bundles. Since the Mostow bundles are not in general principal, the topology of the T-dual backgrounds is largely unexplored. We shall not attempt to do this here, but rather illustrate some of novel features by considering T-duality on the simplest cases of almost abelian manifolds.

Requiring that T-duality preserves supersymmetry imposes that the Lie derivatives with respect to any isometry vector v vanish, $\mathcal{L}_v \Psi_\pm = 0$ [27]. For the simple case of almost abelian solvmanifolds, it is not hard to check that all vectors $v_i = \partial_i$, where, in the basis chosen, $i = 1, ..., 4, 6$, satisfy this condition. However, these vectors are defined only locally[2], since they transform non-trivially under $t \sim t + t_0$. Hence, in general, the result of T-duality will be non-geometric. We shall see that there are subtleties even for the case when the supersymmetry-preserving isometries ∂_i are well defined.

We shall consider the action of T-duality on two solvmanifolds, $\mathfrak{g}_{5.17}^{0,0,\pm1} \times S^1$ (s 2.5) and $\mathfrak{g}_{5.7}^{1,-1,-1} \times S^1$. We mentioned in section 3.3 the existence of solutions on these two manifolds. For s 2.5, following [29], we write the algebra as $(25, -15, r45, -r35, 0, 0)$, $r^2 = 1$. The twist matrix $A(t)$ is made of periodic functions of $t = x^5$,

$$A = \begin{pmatrix} R_{r=1} & & \\ & R_r & \\ & & \mathbb{I}_2 \end{pmatrix} , \qquad R_r = \begin{pmatrix} \cos x^5 & -r \sin x^5 \\ r \sin x^5 & \cos x^5 \end{pmatrix} , \tag{C.2.1}$$

and T-duality is unobstructed. The supersymmetric solutions mentioned in section 3.3 are all related by two T-dualities

$$\begin{array}{ccccccc}
 & \text{IIB} & & & & \text{IIA} & \\
\text{t:30} & & \text{t:12} & & \text{t:30} & & \text{t:12} \\
(13+24) & \longleftrightarrow & (14+23) & \longleftrightarrow & (136+246) & \longleftrightarrow & (146+236) \\
 & \mathrm{T}_{12} & & \mathrm{T}_6 & & \mathrm{T}_{12} & \\
(14+23) & \longleftrightarrow & (13+24) & \longleftrightarrow & (146+236) & \longleftrightarrow & (136+246)
\end{array}$$

In the table we labelled each solution by the dominant O-plane charge. The sources are labelled by their longitudinal directions, e.g. $(13 + 24)$ stands for a solution with two sources (one O5 and one D5) along directions $e^1 \wedge e^3$ and $e^2 \wedge e^4$. T-dualities (the subscripts indicate the directions in which they are performed) exchange the columns in the table; lines are exchanged by relabellings (symmetries of the algebra). The T-dualities are type changing, meaning a pair of type 0 and 3 (t:30) pure spinors is exchanged with a pair of type 1 and 2 (t:12) and vice versa.

[2] As discussed, on the compact solvmanifolds there exists a set of globally defined one forms $\{e\} = \{A_M \mathrm{d}x\}$ and the dual basis $\{E\} = \{(A_M^{-1})^T \partial\}$ is made of globally defined vectors. However, the Lie derivative of the pure spinors with respect to these does not vanish.

It is natural to see what will be the effect of a single T-duality. To be precise we take as starting point Model 3 of [29]. We shall concentrate on the NSNS sector and discuss the topology changes under T-duality. The NSNS flux is zero and the metric, in the $\mathrm{d}x^i$ basis is

$$\begin{aligned} \mathrm{d}s^2 &= \frac{t_1^2}{t_2}(\tau_2^1)^2 G(\mathrm{d}x^1 + \mathcal{A}\mathrm{d}x^2)^2 + \frac{t_1}{G}(\mathrm{d}x^2)^2 + t_1(\tau_2^1)^2 G(\mathrm{d}x^3 + r\mathcal{A}\mathrm{d}x^4)^2 \\ &\quad + \frac{t_2}{G}(\mathrm{d}x^4)^2 + t_3(\mathrm{d}x^5)^2 + t_3(\mathrm{d}x^6)^2 \end{aligned} \tag{C.2.2}$$

with

$$G = \cos^2(x^5) + \frac{t_2}{t_1(\tau_2^1)^2}\sin^2(x^5) \qquad \mathcal{A} = \frac{t_2 - t_1(\tau_2^1)^2}{2G\, t_1(\tau_2^1)^2}\ \sin(2x^5)\,. \tag{C.2.3}$$

A single T-duality along x^1 yields the manifold $T^3 \times \varepsilon_2$ ($\varepsilon_2 : (-23, 13, 0)$) with O6-D4 (or D6-O4) and an H-flux given by

$$H = -\mathrm{d}\mathcal{A} \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\ . \tag{C.2.4}$$

Note that the H-flux (C.2.4) allows for topologically different choices of B-field. Being not completely solvable (see footnote 2 in section 3.2), s 2.5 can yield manifolds of different topology (different Betti numbers). Correspondingly, the results of T-duality should vary as well, and the application of the local Buscher rules might be ambiguous. The choice of B-field in (C.2.4), $B = -\mathcal{A}\mathrm{d}x^1 \wedge \mathrm{d}x^2$, corresponding to the application of the local rules to (C.2.2), is globally defined due to $\mathcal{A}(x^5 + l) = \mathcal{A}(x^5)$. There is a less trivial choice with $B = -x^1\, \partial_5\mathcal{A}\, \mathrm{d}x^2 \wedge \mathrm{d}x^5$ which however does not arise from the application of local T-duality rules to (C.2.2) since the metric does not have off-diagonal elements between x^2 and x^5.

A further T-duality along x^2 gives back s 2.5 with O5-D5 sources, but the supersymmetry now is captured by a different pair of pure spinors.

For the manifold $\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}$, the twist matrix is

$$A(x^5) = \begin{pmatrix} R(x^5) & & \\ & R(-x^5) & \\ & & \mathbb{I}_2 \end{pmatrix}\ , \qquad R(x^5) = \begin{pmatrix} \mathrm{ch} & -\eta_0\mathrm{sh} \\ -\frac{1}{\eta_0}\mathrm{sh} & \mathrm{ch} \end{pmatrix}\ , \tag{C.2.5}$$

where we set

$$\mathrm{ch} = \cosh(\sqrt{q_1 q_2}x^5)\ ,\ \mathrm{sh} = \sinh(\sqrt{q_1 q_2}x^5)\ ,\ \eta_0 = \sqrt{\frac{q_1}{q_2}}\ . \tag{C.2.6}$$

Then it is straightforward to check that the isometry vectors $v_i = \partial_i$ are local. Any T-duality along these is thus obstructed, and hence the O6-D6 solution of [47, 29] does not have geometric T-duals. For this case we shall adopt the method applied to nilmanifolds in [27] and mentioned in section 2.3, and work out the action of T-duality on the generalized vielbein.

The generalized vielbein on $\mathfrak{g}_{5.7}^{1,-1,-1} \oplus \mathbb{R}$ can be obtained using twist transformation (see (4.3.2)) from the generalized vielbein of the torus (on which we take for simplicity the identity metric)

$$\mathcal{E} = \left(\begin{array}{c|c} \mathbb{I}_6 & 0_6 \\ \hline 0_6 & \mathbb{I}_6 \end{array}\right) \left(\begin{array}{c|c} A & 0_6 \\ \hline 0_6 & A^{-T} \end{array}\right)\ . \tag{C.2.7}$$

To work out their T-duals, we act by

$$\mathcal{E}_T = O_T \times \mathcal{E} \times O_T\ , \tag{C.2.8}$$

where O_T is the $\mathrm{O}(d,d)$ matrix for T-duality (see (2.3.16)). The O_T on the right is the regular action of T-duality, while the O_T on the left assures that the map has no kernel (see [27]). The T-duality is done in the x^1 direction, so the O_T is

$$O_T = \left(\begin{array}{ccc|ccc} T_1 & & & T_2 & & \\ & \mathbb{I}_2 & & & 0_2 & \\ & & \mathbb{I}_2 & & & 0_2 \\ \hline T_2 & & & T_1 & & \\ & 0_2 & & & \mathbb{I}_2 & \\ & & 0_2 & & & \mathbb{I}_2 \end{array}\right) \ , \ T_1 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \ , \ T_2 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \ , \tag{C.2.9}$$

and then

$$\mathcal{E}_T = \left(\begin{array}{ccc|ccc} C_1 & & & B_1 & & \\ & R(-x^5) & & & 0_2 & \\ & & \mathbb{I}_2 & & & 0_2 \\ \hline B_2 & & & C_2 & & \\ & 0_2 & & & R(x^5)^T & \\ & & 0_2 & & & \mathbb{I}_2 \end{array}\right) \ , \tag{C.2.10}$$

with

$$C_1 = C_2 = \text{ch } \mathbb{I}_2 \ , \ B_1 = -\frac{1}{\eta_0}\text{sh } \epsilon \ , \ B_2 = \eta_0 \text{sh } \epsilon \ , \ \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ . \tag{C.2.11}$$

The generalized vielbein $\mathcal{E}_T$ can be brought to the canonical lower diagonal form (2.3.23) by a local $O(d) \times O(d)$ transformation. When such a transformation cannot be made single-valued, we talk about non-geometric backgrounds (where the action of a non-trivial β cannot be gauged away). The result of the $O(d) \times O(d)$ transformation is

$$\mathcal{E}' = \left(\begin{array}{ccc|ccc} O_1 & & & O_2 & & \\ & \mathbb{I}_2 & & & 0_2 & \\ & & \mathbb{I}_2 & & & 0_2 \\ \hline O_2 & & & O_1 & & \\ & 0_2 & & & \mathbb{I}_2 & \\ & & 0_2 & & & \mathbb{I}_2 \end{array}\right) \times \mathcal{E}_T = \left(\begin{array}{ccc|ccc} O_1C_1 + O_2B_2 & & & O_1B_1 + O_2C_2 & & \\ & R_2 & & & 0_2 & \\ & & \mathbb{I}_2 & & & 0_2 \\ \hline O_2C_1 + O_1B_2 & & & O_2B_1 + O_1C_2 & & \\ & 0_2 & & & R_2^{-T} & \\ & & 0_2 & & & \mathbb{I}_2 \end{array}\right) \ , \tag{C.2.12}$$

where the non-trivial $O(d) \times O(d)$ components are (see 2.3.29)

$$O_{1/2} = \frac{1}{2}(O_+ \pm O_-) \qquad O_\pm \in O(2) \ . \tag{C.2.13}$$

By solving $O_1B_1 + O_2C_2 = 0$, we can obtain O_2 and express $O_\pm$ in terms of O_1:

$$\begin{aligned} O_\pm &= O_1(\mathbb{I}_2 \pm u\ \epsilon) \ , \quad u = \frac{\text{sh}}{\eta_0 \text{ch}} \ , \\ O_\pm^T O_\pm &= \mathbb{I}_2 \quad \Leftrightarrow \quad O_1^T O_1 = \frac{1}{1+u^2}\ \mathbb{I}_2 \ . \end{aligned} \tag{C.2.14}$$

A simple solution is given by

$$O_1 = \frac{1}{\sqrt{1+u^2}}\ \mathbb{I}_2 \quad \Rightarrow \quad O_2 = \frac{u}{\sqrt{1+u^2}}\ \epsilon \ . \tag{C.2.15}$$

Thus we can indeed bring $\mathcal{E}_T$ to a lower-diagonal form, but with an $O(d) \times O(d)$ transformation that is not globally defined. It is not hard to see that replacing the x^1 direction by others does not change much. Hence any T-dual to $\mathfrak{g}_{5.7}^{1,-1,-1} \times S^1$ is non-geometric.

A similar analysis for s 2.5 shows that one can easily solve the constraint $O_1B_1 + O_2C_2 = 0$ with O_1 and O_2 being globally defined (this is easy since the functions entering are all periodic).

C.3 Transforming the gauge bundle in heterotic compactifications

C.3.1 New conventions and transformations considered

As discussed at the end of section 4.5, in order to map the gauge fields $\mathcal{F}$ of the two heterotic solutions considered, we should extend our transformation on the generalized tangent bundle (and generalized vielbein) to the gauge bundle. T-duality and $O(n,n)$ transformations in heterotic string have been extended to the gauge bundle by considering $O(n+16,n+16)$ transformations. These were introduced in [99, 100]. We will follow the same procedure and extend our $O(d,d)$ transformation to $O(d+16,d+16)$. Basically, we have to extend every matrix considered so far by 16 complex components to get them on a dimension $d+16$ bundle. So we define these extended quantities:

$$e = \begin{pmatrix} e_s & 0 \\ e_g\mathcal{A} & e_g \end{pmatrix}, \quad g = e^T e = \begin{pmatrix} g_s + \mathcal{A}^T g_g \mathcal{A} & \mathcal{A}^T g_g \\ g_g\mathcal{A} & g_g \end{pmatrix}, \quad B = \begin{pmatrix} B_s & -\mathcal{A}^T g_g \\ g_g\mathcal{A} & B_g \end{pmatrix}, \tag{C.3.1}$$

where the s index denotes the space-time objects (they are the same as in section 4.3.1), and the g index denotes the gauge bundle quantities. $\mathcal{A}$ is the $16\times d$ matrix giving the gauge connection. $g_g = e_g^T e_g$ and B_g are the "gauge" metric and B-field, which are actually constrained to take specific values, in order to make sense with the (root) lattice on which we consider the fields

$$g_g = \frac{1}{2}\mathcal{C}, \quad (B_g)_{ij} = \begin{cases} -(g_g)_{ij} & i<j \\ 0 & i=j \\ (g_g)_{ij} & i>j \end{cases} \tag{C.3.2}$$

where $\mathcal{C}$ is the Cartan matrix (symmetric) of the group considered. As these matrices are fixed, the only new freedom we introduce is the gauge connection given by $\mathcal{A}$.

Then we define as before the generalized metric $\mathcal{H}$ and the generalized vielbein $\mathcal{E}$, which are now extended to the gauge bundle:

$$\tilde{\mathcal{E}} = \begin{pmatrix} e & 0 \\ -\hat{e}\,B & \hat{e} \end{pmatrix} \qquad \mathcal{H} = \tilde{\mathcal{E}}^T\,\tilde{\mathcal{E}} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}, \tag{C.3.3}$$

and are therefore $2(d+16)\times 2(d+16)$ matrices. The $O(d+16,d+16)$ transformations act on them as did $O(d,d)$ on the generalized vielbein and metric, (2.3.25) and (2.3.22). We define the transformation of the dilaton as before (2.3.34); as we will see, we can use either the previous $d\times d$ metric or the new $(d+16)\times(d+16)$ one.

As in [99, 100], we shall consider a subset[3] of $O(d+16,d+16)$, which does not change e_g and B_g. Indeed, e_g and B_g are related to the Cartan matrix which should stay invariant. Furthermore, the transformation should preserve the off-diagonal structure of B, i.e. the off-diagonal block of the transformed B should be related in the same way to the new gauge connection.

Following the logic of section 4.3.1, we consider the following $O(d+16,d+16)$ transformations

$$O = \begin{pmatrix} A & 0 \\ C & A^{-T} \end{pmatrix}, \tag{C.3.4}$$

which satisfies the $O(d+16,d+16)$ constraint $A^TC + C^TA = 0_{d+16}$, and where, according to (C.3.1), the matrices A and C can be decomposed into space and gauge blocks

$$A = \begin{pmatrix} A_s & 0 \\ A_o & A_g \end{pmatrix}, \ C = \begin{pmatrix} C_s & C_o \\ C_o' & C_g \end{pmatrix}. \tag{C.3.5}$$

[3] One can show this subset is a subgroup of $O(d+16,d)$, because it preserves the last $(d+16)\times 16$ block column of $g+B$.

The transformed vielbein read

$$\tilde{\mathcal{E}}' = \begin{pmatrix} e' & 0 \\ -\hat{e}' B' & \hat{e}' \end{pmatrix} , \quad e' = eA, \quad B' = A^T B A - A^T C \ . \tag{C.3.6}$$

Imposing the invariance of the e_g component of the vielbein sets $A_g = \mathbb{I}_{16}$ and gives the new gauge connection $\mathcal{A}' = \mathcal{A} A_s + A_o$. Similarly the invariance of B_g in the B-field implies $C_g = 0_{16}$. Then we have to ask that the off-diagonal terms in B can be written again in the form (C.3.1). This fixes C_o and C'_o

$$C_o = A_s^{-T} A_o^T (g_g + B_g) \qquad C'_o = (B_g - g_g) A_o \ . \tag{C.3.7}$$

Finally it is easy to see that the $O(d+16, d+16)$ constraint $A^T C + C^T A = 0_{d+16}$ is equivalent to the antisymmetry of transformed B-field in (C.3.6) and gives the constraint

$$A_s^T C_s + C_s^T A_s = 2 A_o^T g_g A_o \ . \tag{C.3.8}$$

Out of all these constraints, the only remaining degrees of freedom of the transformation are then A_s, A_o, and C_s, which are constrained with (C.3.8). With respect to the $O(d,d)$ transformation, we gain A_o, which can act on the gauge connection (the only new freedom introduced in $\tilde{\mathcal{E}}$).

C.3.2 A specific case: the Kähler/non-Kähler transition of section 4.5

Let us now focus on the specific examples we considered in section 4.5. Solution 1 is a trivial T^2 fibration, with no B-field, so we set $B_s = 0$, and has a non-trivial gauge connection $\mathcal{A} \neq 0$. To recover Solution 2, we want to produce a connection in the metric, a non-zero B-field, and no gauge connection, i.e. $\mathcal{A}' = 0$. From section 4.3.1, it is easy to write the metric part of the transformation A

$$A_s = \begin{pmatrix} \mathbb{I}_4 & 0 \\ A_{\mathcal{C}} & \mathbb{I}_2 \end{pmatrix} . \tag{C.3.9}$$

Since the diagonal elements are just identity matrices, this transformation does not modify the metric and the dilaton. The vanishing of the gauge field $\mathcal{A}' = 0$ simply tells us to choose $A_o = -\mathcal{A} A_s$. So the choice of connections fixes completely the A matrix.

We have now to check whether the constraint (C.3.8) can be satisfied. If we take the gauge connection in Solution 1 to be only on the base, the off-diagonal block in the vielbein (C.3.1) takes the form $e_g \mathcal{A} = \begin{pmatrix} \mathcal{A}_{\mathcal{B}} & 0_{16\times 2} \end{pmatrix}$, then the constraint (C.3.8) becomes

$$A_s^T C_s + C_s^T A_s = 2 \mathcal{A}^T g_g \mathcal{A} \ , \tag{C.3.10}$$

and it is easy to verify that it solved by the following choice for the matrix C_s

$$C_s = \begin{pmatrix} \tilde{C}_{\mathcal{B}} - A_{\mathcal{C}}^T C_{\mathcal{C}} + \mathcal{A}_{\mathcal{B}}^T \mathcal{A}_{\mathcal{B}} & -(C_{\mathcal{C}}^T + A_{\mathcal{C}}^T C_{\mathcal{F}}) \\ C_{\mathcal{C}} & C_{\mathcal{F}} \end{pmatrix} , \tag{C.3.11}$$

where $\tilde{C}_{\mathcal{B}}$, $C_{\mathcal{F}}$ and $C_{\mathcal{C}}$ are free, and the two first are antisymmetric. Note the new B-field is then given by

$$B'_s = - \begin{pmatrix} \tilde{C}_{\mathcal{B}} & -C_{\mathcal{C}}^T \\ C_{\mathcal{C}} & C_{\mathcal{F}} \end{pmatrix} \tag{C.3.12}$$

so we see once again that we can choose it to be whatever we want, and it fixes completely the C matrix.

To summarise, inspired by the T-duality in heterotic strings we have made some steps towards extending the $O(d,d)$ generalized tangent bundle transformations to $O(d+16, d+16)$ hence covering the transformations of the gauge bundle. This allows, in particular, to relate the two solutions discussed in section 4.5.

Appendix D

Résumé long en français

Les théories de cordes sont des candidates très intéressantes en vue d'une théorie quantique de la gravitation. Tâcher d'en faire des théories unificatrices des interactions fondamentales a donc été considéré depuis longtemps. Pour y parvenir, elles devraient pouvoir reproduire à basse énergie le modèle standard de la physique des particules. Réussir à faire ce lien n'est pas si simple, car les théories de cordes sont dotés de plusieurs ingrédients supplémentaires non observés. En particulier, la plupart d'entre elles sont définies à dix dimensions d'espace-temps, et sont supersymétriques. Dans le schéma habituel pour retrouver la physique que l'on connaît, la supersymétrie est préservée pour des arguments phénoménologiques ou simplement techniques, et l'on essaye plutôt de reproduire une extension supersymétrique du modèle standard. Par contre, les six dimensions d'espace supplémentaires ne peuvent être conservées. On considère alors qu'elles ne sont pas étendues, mais qu'elles forment un espace compact (par exemple six cercles) de taille suffisamment petite pour ne pas avoir été détecté par nos expériences. Il existe de nombreuses possibilités pour le choix de cet espace compact à six dimensions, appelé espace interne. Mathématiquement, il s'agit d'une variété différentielle M, et le choix de ses propriétés (sa topologie, etc.) va avoir une importance capitale. Il existe une procédure dite de réduction dimensionnelle pour passer d'une théorie à dix dimensions à une théorie à quatre dimensions. Dans cette procédure, les caractéristiques de M ont une influence importante sur la théorie résultante à quatre dimensions. Par conséquent, des critères phénoménologiques, comme la préservation de la supersymétrie, ou l'absence de champs scalaires non massifs, vont alors être utilisés pour contraindre le choix de cette variété M.

Pour relier les théories de cordes à des théories de basse énergie quatre-dimensionnelles, on part tout d'abord d'une des théories de supergravité dix-dimensionnelles. Celles-ci sont les théories de basse énergie des théories de cordes. On cherche ensuite une solution dix-dimensionnelle à cette théorie. Puis on doit déterminer les modes légers (fluctuations) autour de cette solution. Les degrés de liberté de la théorie sont alors tronqués à ces modes légers. On peut ensuite effectuer la réduction dimensionnelle qui consiste à intégrer sur les degrés de libertés internes. On obtient une théorie effective de basse énergie à quatre dimensions. Dans cette thèse, nous allons seulement nous focaliser sur une étape de ce programme, qui consiste à trouver et à étudier les solutions de la supergravité dix-dimensionnelle, sur des variétés internes potentiellement intéressantes pour la phénoménologie.

Nous commencerons par une présentation générale des solutions préservant la supersymétrie, et reviendrons sur la question du choix de la variété interne, puis nous motiverons l'utilisation du formalisme mathématique de Géométrie Complexe Généralisée. Etant donné ce contexte, nous discuterons alors plus en détails le travail effectué en thèse et les résultats obtenus.

D.1 Solutions supersymétriques et Géométrie Complexe Généralisée

Une partie importante de cette thèse est consacrée à l'étude de solutions supersymétriques de la supergravité de type II avec flux non-triviaux, dans le cadre de compactifications sur un espace six-dimensionnel. Les conditions pour avoir un vide (une solution) supersymétrique imposent des contraintes sur la géométrie de la variété interne M. Dans le cas connu de compactifications sans flux, avoir une supersymétrie minimale impose que la variété interne soit un Calabi-Yau. En présence de flux, les conditions de supersymétrie peuvent se réécrire de manière simple [4, 5] en utilisant le formalisme de Géométrie Complexe Généralisée, récemment développé par Hitchin et Gualtieri [6, 7]. La variété interne est alors caractérisée comme étant un Calabi-Yau Généralisé.

Nous allons tout d'abord donner les ingrédients dont nous aurons besoin pour chercher des vides avec flux. Nous présentons nos conventions pour la supergravité, dans le contexte d'une compactification vers quatre dimensions. Ensuite, nous discutons la préservation de la supersymétrie et motivons l'utilisation de la Géométrie Complexe Généralisée, en particulier de la notion de spineurs purs. Nous présentons enfin les conditions pour préserver la supersymétrie réécrites en termes de ces spineurs purs.

D.1.1 Solutions de supergravité et variété interne

Supergravité en quatre plus six dimensions

Dans cette thèse, nous sommes intéressés par les compactifications vers quatre dimensions, où l'espace-temps quatre-dimensionnel est maximalement symétrique: Minkowski, de Sitter ou Anti de Sitter. Nous allons donc présenter la supergravité de type IIA ou IIB en proposant certaines formes pour ses champs, qui respectent cette séparation de l'espace-temps dix-dimensionnel.

Nous considérerons un espace-temps dix-dimensionnel qui est un produit conforme d'un espace-temps quatre-dimensionnel maximalement symétrique, et d'un espace six-dimensionnel compact M (espace interne). Par conséquent, la métrique dix-dimensionnelle est de la forme

$$ds^2_{(10)} = e^{2A(y)}\ g_{\mu\nu}dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n \ , \tag{D.1.1}$$

où e^{2A} est le facteur conforme dépendant des dimensions internes y^m. La métrique quatre-dimensionnelle, de signature $(-1,+1,+1,+1)$, a comme groupe de symétrie le groupe de Poincaré, $SO(1,4)$ ou $SO(2,3)$, pour M_4, AdS_4 ou dS_4 respectivement. Le dilaton ϕ va souvent être relié au facteur conforme.

Pour les flux de RR et NSNS, nous pouvons leur permettre a priori d'avoir des valeurs non-nulles dans le vide. Néanmoins, la symétrie maximale quatre-dimensionnelle impose d'avoir des flux non-triviaux seulement sur la variété interne. Le potentiel de jauge B du secteur NSNS et son tenseur de Faraday $H = \mathrm{d}B$ sont donc purement internes, et les tenseurs de Faraday des flux RR dix-dimensionnels $F_k^{(10)}$ sont restreints à l'ansatz suivant, en terme des flux internes F_k

$$F_k^{(10)} = F_k + \mathrm{vol}_4 \wedge \lambda(*F_{6-k})\,. \tag{D.1.2}$$

Ici, $*$ est le "Hodge star" six-dimensionnel, $\mathrm{vol}_{(4)}$ est le volume conforme quatre-dimensionnel, et λ agit sur une p-forme A_p par une inversion complète de ses indices

$$\lambda(A_p) = (-1)^{\frac{p(p-1)}{2}} A_p \ . \tag{D.1.3}$$

On définit le champ RR interne total F par

$$\mathrm{IIA} \ : \quad F = F_0 + F_2 + F_4 + F_6 \ , \tag{D.1.4}$$

$$\mathrm{IIB} \ : \quad F = F_1 + F_3 + F_5 \ , \tag{D.1.5}$$

où F_k est la k-forme RR interne. Ainsi, les équations du mouvement des flux et leur identités de Bianchi s'expriment comme

$$\begin{aligned}
&\mathrm{d}(e^{4A-2\phi} * H) \pm e^{4A} \sum_k F_k \wedge *F_{k+2} = \text{terme de source} \ , \qquad && \mathrm{d}H = 0 \ , \\
&(\mathrm{d} + H\wedge)(e^{4A} * F) = 0 \ , && (\mathrm{d} - H\wedge)F = \delta_s \ ,
\end{aligned} \tag{D.1.6}$$

où le signe haut/bas est pour IIA/B, et d et $*$ sont internes. Dans les identités de Bianchi, on suppose l'absence de source NSNS, et δ_s indique la contribution des sources RR. Les compactifications avec flux vers Minkowski ou de Sitter à quatre dimensions ne sont possibles qu'avec des sources à tension négative, les orientifolds (notés Op pour un objet étendu dans p dimensions d'espace). Ces sources sont nécessaires pour annuler la contribution positive des flux à la trace du tenseur d'énergie-impulsion [12, 13]. De plus, pour ne pas briser la symétrie maximale à quatre dimensions, nous considérerons seulement des D-branes ou des orientifolds étendus dans tout l'espace quatre-dimensionnel.

Le secteur fermionique de la supergravité est constitué d'un doublet de gravitino ψ^i_M (M indice d'espace-temps dix-dimensionnel) et d'un doublet de dilatino $\tilde{\lambda}^i$. Imposant la symétrie maximale, leur valeur dans le vide doit être zéro. Nous chercherons donc des solutions purement bosoniques.

Solutions supersymétriques

Nous allons restreindre davantage la forme de nos solutions en imposant qu'elles préservent une supersymétrie minimale $\mathcal{N} = 1$ à quatre dimensions. D'un point de vue physique, cela signifie que la supersymétrie devra être brisée à plus basse énergie. D'un point de vue technique, chercher des solutions supersymétriques simplifie grandement la résolution. En effet, pour les solutions de Minkowski, il a été prouvé [14, 15, 11] que toutes les équations du mouvement sont automatiquement satisfaites dès lors que les conditions de supersymétrie, et les identités de Bianchi pour les flux sont vérifiées. Par conséquent, au lieu de résoudre les équations du mouvement qui peuvent être des équations du second ordre, il suffit de résoudre des équations du premier ordre, ce qui est plus simple.

Pour un vide purement bosonique, les conditions pour préserver la supersymétrie sont données par l'annulation des variations supersymétriques des champs fermioniques. En effet, ces variations sont bosoniques donc ne sont pas automatiquement nulles dans le vide. Cela signifie en supergravité de type II que l'on doit annuler les variations supersymétriques du gravitino et du dilatino

$$\delta\psi_M = 0 \ , \qquad \delta\tilde{\lambda} = 0 \, . \tag{D.1.7}$$

Elles s'écrivent

$$\delta\psi_M = (D_M + \frac{1}{4} H_M \mathcal{P})\epsilon + \frac{1}{16} e^{\phi} \sum_n \not{F}^{(2n)} \, \Gamma_M \mathcal{P}_n \, \epsilon \, , \tag{D.1.8}$$

$$\delta\tilde{\lambda} = (\not{\partial}\phi + \frac{1}{2} \not{H} \mathcal{P})\epsilon + \frac{1}{8} e^{\phi} \sum_n (-1)^{2n} (5 - 2n) \not{F}^{(2n)} \, \mathcal{P}_n \epsilon \, , \tag{D.1.9}$$

où le paramètre de supersymétrie $\epsilon = (\epsilon^1, \epsilon^2)$ est un doublet de spineurs Majorana-Weyl. Les matrices $\mathcal{P}$ et $\mathcal{P}_n$ sont différentes en IIA et IIB. En IIA $\mathcal{P} = \Gamma_{11}$ et $\mathcal{P}_n = \Gamma_{11}\sigma_1$, tandis qu'en IIB $\mathcal{P} = -\sigma_3$, $\mathcal{P}_n = \sigma_1$ pour $n + 1/2$ pair et $i\sigma_2$ pour $n + 1/2$ impair.

Dans notre ansatz pour la métrique et les flux, nous avons pris en compte que l'espace-temps se décomposait en quatre plus six dimensions. Les paramètres de supersymétrie ϵ^1 et ϵ^2 doivent se décomposer de la même manière. Le groupe de Lorentz est brisé en $SO(1,3) \times SO(6)$ donc les paramètres s'écrivent comme un produit sous les représentations spinorielles de ces groupes. Pour un vide $\mathcal{N} = 1$, on considère à quatre dimensions un seul paramètre spinoriel chiral ζ_+. Il faut alors une paire (η^1, η^2) de spineurs de Weyl à six dimensions. Etant donné les chiralités des deux théories, on considère donc la décomposition suivante en IIA

$$\begin{aligned} \epsilon^1 &= \zeta_+^1 \otimes \eta_+^1 + \zeta_-^1 \otimes \eta_-^1 \ , \\ \epsilon^2 &= \zeta_+^2 \otimes \eta_-^2 + \zeta_-^2 \otimes \eta_+^2 \ , \end{aligned} \tag{D.1.10}$$

et celle-ci en IIB

$$\epsilon^{i=1,2} = \zeta_+^i \otimes \eta_+^i + \zeta_-^i \otimes \eta_-^i \ , \tag{D.1.11}$$

où la conjugaison complexe change la chiralité: $(\eta_+)^* = \eta_-$. Pour un vide $\mathcal{N} = 1$, on prend $\zeta_+^1 = \zeta_+^2 = \zeta_+$.

En général, on demande que les spineurs internes soient en plus globalement définis (ne s'annulant jamais). La justification pour une telle hypothèse vient de la réduction dimensionnelle: pour avoir une théorie supersymétrique à quatre dimensions, on a besoin d'une base de spineurs internes globalement définis pour pouvoir réduire les paramètres de supersymétrie. Cette contrainte topologique va jouer un rôle important. Elle peut tout d'abord être traduite en terme de G-structures.

On dit qu'une variété M admet une G-structure lorsque le groupe de structure du fibré tangent est G. Ce groupe correspond au groupe des fonctions de transitions. A six dimensions, il s'agit a priori de $GL(6)$. L'existence de tenseurs ou de spineurs globalement définis conduit à une réduction de ce groupe. Par exemple, étant donné une métrique et une orientation, il est réduit à $SO(6) \sim SU(4)$. En présence d'un spineur globalement défini, il est réduit davantage en $SU(3)$, et même en $SU(2)$ dans le cas d'un second spineur indépendant et également globalement défini.

A six dimensions, étant donné un spineur de Weyl de chiralité positive et de norme unitaire η_+ globalement défini, on peut définir de manière équivalente la structure $SU(3)$ en terme de formes invariantes. On définit la trois-forme holomorphe Ω et la forme de Kähler J comme

$$\begin{aligned} \Omega_{\mu\nu\rho} &= -i\eta_-^\dagger \gamma_{\mu\nu\rho}\eta_+ \ , \\ J_{\mu\nu} &= -i\eta_+^\dagger \gamma_{\mu\nu}\eta_+ \ . \end{aligned} \tag{D.1.12}$$

De même, pour une structure $SU(2)$ définie par deux spineurs orthogonaux globalement définis η_+ et χ_+ de norme unitaire, on peut définir des formes invariantes: une un-forme holomorphe z (que l'on prend de norme $||z||^2 = 2$), une deux-forme réelle j, et une deux-forme holomorphe ω

$$\begin{aligned} z_\mu &= \eta_-^\dagger \gamma_\mu \chi_+ \ , \\ j_{\mu\nu} &= -i\eta_+^\dagger \gamma_{\mu\nu}\eta_+ + i\chi_+^\dagger \gamma_{\mu\nu}\chi_+ \ , \\ \omega_{\mu\nu} &= \eta_-^\dagger \gamma_{\mu\nu}\chi_- \ . \end{aligned} \tag{D.1.13}$$

Dans les deux cas, ces formes doivent satisfaire certaines conditions pour définir la G-structure.

La contrainte topologique donnée par l'existence de deux spineurs internes globalement définis $\eta_+^{i=1,2}$ est équivalente à l'existence de formes globalement définies, qui satisfont des conditions dites de structure. Comme on va le voir, ces formes offrent également une manière alternative d'exprimer les contraintes différentielles pour la préservation de la supersymétrie, données par l'annulation des variations supersymétriques des fermions. Comme premier exemple, nous allons tout d'abord discuter le cas des compactifications sans flux, où la préservation de la supersymétrie amène à la condition de Calabi-Yau.

Cas des variétés de Calabi-Yau

Considérons des solutions sans flux: $F_k = 0$, $H = 0$. On suppose également que la variété n'admet qu'un seul spineur globalement défini $\eta_+^1 = \eta_+^2 = \eta_+$. Dans ce cas, on obtient a priori une théorie

$\mathcal{N}=2$ à quatre dimensions avec deux spineurs quatre-dimensionnels $\zeta^{1,2}$. Avec cette hypothèse, en décomposant les spineurs et les conditions de supersymétrie (D.1.8) et (D.1.9) en parties quatre et six-dimensionnelles, on obtient

$$\partial_\mu \zeta_+ = 0 \ , \ D_m \eta^1_+ = 0 \ , \tag{D.1.14}$$

où m est un indice interne. Cela signifie que la variété interne doit admettre non seulement un spineur globalement défini (contrainte topologique), mais celui-ci doit être constant de manière covariante (contrainte différentielle). Cela implique que le groupe d'holonomie de M est réduit à $SU(3)$, et par conséquent la variété doit être un Calabi-Yau [16].

En terme de G-structures, on obtient une structure $SU(3)$. De plus, la fermeture du spineur se traduit en conditions différentielles sur les formes définissant la structure $SU(3)$. Celles-ci doivent être fermées:

$$\mathrm{d}J = 0 \ , \ \mathrm{d}\Omega = 0 \ . \tag{D.1.15}$$

Ces conditions donnent l'intégrabilité des structures quasi-complexe et symplectique, ce qui signifie que la variété interne doit être Kähler. C'est en effet le cas d'un Calabi-Yau. Notez qu'une autre propriété des variétés de Calabi-Yau est d'être plate.

En l'absence de flux, chercher un vide supersymétrique demande donc de considérer une variété interne étant un Calabi-Yau. Par réduction dimensionnelle de l'action de type II sur un Calabi-Yau, on obtiendra une théorie effective $\mathcal{N}=2$ à quatre dimensions (une réduction similaire en corde hétérotique mènerait à une théorie $\mathcal{N}=1$). Nous n'allons pas présenter en détail cette réduction mais seulement insister sur deux aspects. Tout d'abord, la géométrie de la variété interne apparaît donc comme cruciale pour déterminer les symétries et le contenu en champs de la théorie effective quatre-dimensionnelle. De plus, les théories effectives construites par réduction sur Calabi-Yau souffrent toutes de la présence de champs scalaires sans masse, non contraints par un quelconque potentiel. Ils sont appelés moduli. En théorie supersymétrique, ils ne posent a priori pas de problème particulier. Cependant, si certains d'entre eux demeurent sans masse après la brisure de la supersymétrie, cela pose un problème phénoménologique: des champs scalaires sans masse seraient porteurs d'interactions à longue portée qui devraient être observées (mis à part le cas d'un scénario de confinement).

On cherche donc des mécanismes qui permettraient de stabiliser au moins quelques-uns de ces moduli au niveau de la théorie supersymétrique. Supposons qu'un champ scalaire φ apparaissant dans la théorie effective est soumis à un potentiel $V(\varphi)$. Si ce potentiel admet un minimum en une valeur φ_0, alors l'action et le potentiel peuvent être développés autour de ce minimum

$$V(\varphi) \approx V(\varphi_0) + V''(\varphi_0)(\partial\varphi)^2 \ . \tag{D.1.16}$$

Par conséquent, donner une valeur dans le vide à un champ scalaire (le "fixer") lui donne une masse (un terme de masse). Pour peu que cette masse soit suffisamment élevée, on peut intégrer sur le champ scalaire et ainsi s'en débarrasser. Comme mentionné, les compactifications sur Calabi-Yau ne génèrent malheureusement pas de potentiel pour les scalaires.

Vides avec flux non-triviaux

Ce problème des moduli a conduit dans les années 2000 au développement des compactifications avec flux: on cherche des solutions en présence de flux à valeur non-triviale dans le vide. De tels flux sur la variété interne sont intéressant car ils génèrent un potentiel qui peut fixer quelques moduli, si ce n'est tous (dans le cas de certaines compactifications sur AdS). Les moduli restant sont fixés la plupart du temps par des corrections non-perturbatives. Voir [17] pour des revues sur le sujet.

La présence de flux change drastiquement les propriétés des solutions. Typiquement, les flux vont courber la variété interne via leur densité d'énergie, et par conséquent, M ne peut a priori plus être plate. En particulier, la variété interne n'est plus un Calabi-Yau. La présence des flux modifie les conditions de supersymétrie, comme on peut le voir dans (D.1.8) et (D.1.9). Sur l'espace interne, on obtient typiquement des membres de droite dans (D.1.14) ou (D.1.15) qui sont non-nuls, et dépendent des flux. Par exemple, les composantes internes pour le gravitino deviennent

$$\delta\psi^1_m = (D_m + \frac{1}{4}H_m)\eta^1_+ + \not{F}_m\eta^1_+ + \not{F}_m\eta^2_+ \,, \tag{D.1.17}$$

$$\delta\psi^2_m = (D_m + \frac{1}{4}H_m)\eta^2_+ + \not{F}_m\eta^2_+ + \not{F}_m\eta^1_+ \,. \tag{D.1.18}$$

L'exemple le plus ancien de vide avec flux est celui des solutions de cordes hétérotiques avec H non-nul [18, 19]. Dans ce contexte, en présence de flux, la variété n'est plus que complexe, car J n'est désormais plus fermée (voir (D.1.15)). Pour certaines solutions avec flux, la variété ne diffère pas énormément d'un Calabi-Yau: elle peut n'être qu'un Calabi-Yau conforme, où le facteur conforme peut tendre vers 1 dans une certaine limite. Mais dans d'autres cas, comme dans l'exemple de la corde hétérotique, la déviation au Calabi-Yau peut être plus radicale: la topologie peut changer, ce qui rend une limite éventuelle au Calabi-Yau impossible. L'exemple typique est le tore twisté: une fibration non-triviale de cercles sur une base étant un tore. Nous allons utiliser certains d'entre eux (les variétés nilpotentes, et résolubles) et les étudier plus en détails.

Il est donc naturel de demander si l'on peut donner préciser davantage la géométrie de la variété interne, en présence de flux. Une caractérisation mathématique de cette variété a été donnée en supergravité de type II pour les compactifications vers Minkowski [4, 5]: en présence de flux, la variété interne doit être un Calabi-Yau Généralisé. Cette définition prend son sens dans le formalisme de la Géométrie Complexe Généralisée, développée récemment par Hitchin et Gualtieri [6, 7]. La condition de Calabi-Yau Généralisée provient d'une réécriture des conditions différentielles pour préserver la supersymétrie en présence des flux, en terme d'objets de Géométrie Complexe Généralisée: les spineurs purs. Cette condition sur M n'est malheureusement que nécessaire et non suffisante, à la différence du cas du Calabi-Yau. Comme on va le voir, les contraintes supplémentaires proviennent des flux de RR, qui ne sont pas véritablement incorporés dans ce formalisme.

La Géométrie Complexe Généralisée permet de décrire des variétés qui sont complexes, symplectiques, ou partiellement complexes et partiellement symplectiques. On peut définir une structure plus générale, nommée structure complexe généralisée, qui incorpore tous les cas précédents en un seul formalisme, ce qui aide à comprendre la zoologie des variétés apparaissant en présence de flux. Par exemple, les variétés nilpotentes, même si elles n'admettent pas toujours une structure complexe ou symplectique, sont toutes des Calabi-Yau Généralisés (un sous-cas de Complexe Généralisé) [23]. Donc ces variétés seront particulièrement intéressantes pour nous.

Enfin, notez que le formalisme de la Géométrie Complexe Généralisée a l'avantage d'incorporer de manière naturelle une action $O(6,6)$ qui inclue le groupe de T-dualité, et pourrait donc jouer un rôle plus large que celui que nous avons mentionné.

D.1.2 Spineurs purs de Géométrie Complexe Généralisée et vides de supergravité

Jusqu'à présent, nous avons discuté la possibilité d'obtenir des solutions supersymétriques de la supergravité de type II, sur un espace-temps dix-dimensionnel séparé en un espace-temps quatre-dimensionnel maximalement symétrique, et un espace interne six-dimensionnel. En présence de flux, utiles pour résoudre le problème des moduli, nous avons motivé l'introduction du formalisme de la Géométrie Complexe Généralisée. Nous allons à présent donner plus de détails sur l'utilisation de ce formalisme.

La Géométrie Complexe Généralisée considère le fibré tangent généralisé E. Pour une variété M de dimension d, il s'agit de la fibration a priori non-triviale de l'espace cotangent T^*M sur l'espace tangent TM:

$$\begin{array}{ccc} T^*M & \hookrightarrow & E \\ & & \downarrow \\ & & TM \end{array} \tag{D.1.19}$$

Les sections de E sont des vecteurs généralisés et peuvent être écrit localement comme la somme d'un vecteur et d'une un-forme

$$V = v + \xi = \begin{pmatrix} v \\ \xi \end{pmatrix} \in TM \oplus T^*M \, . \tag{D.1.20}$$

On peut également considérer localement des spineurs sur $TM \oplus T^*M$. Ces spineurs vont nous permettre de réexprimer les conditions pour préserver la supersymétrie.

Spineurs internes et spineurs purs de Géométrie Complexe Généralisée

Considérons des spineurs sur $TM \oplus T^*M$. Ce sont des spineurs Cliff(d,d) de Majorana-Weyl, et on peut les voir comme des polyformes, c'est-à-dire des sommes de formes différentielles paires/impaires, qui correspondent à des spineurs de chiralité positive/négative. On va s'intéresser à des spineurs purs: ce sont des vides de Cliff(d,d). Un spineur de Cliff(d,d) est pur si il est annihilé par la moitié des matrices gamma cette algèbre.

Les spineurs purs de Cliff$(6,6)$ sur $TM \oplus T^*M$ peuvent être obtenus comme produit tensoriel de spineurs de Cliff(6), car les bispineurs sont isomorphes aux formes via la relation de Clifford:

$$C = \sum_k \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \quad \leftrightarrow \quad C = \sum_k \frac{1}{k!} C^{(k)}_{i_1 \ldots i_k} \gamma^{i_1 \ldots i_k} \ , \tag{D.1.21}$$

et à six dimensions, tout spineur de Cliff(6) est pur. Dans le contexte de la supergravité, il est donc naturel de définir les spineurs (purs) de Cliff$(6,6)$ sur $TM \oplus T^*M$ comme un produit des paramètres internes de supersymétrie

$$\begin{aligned} \Phi_+ &= \eta^1_+ \otimes \eta^{2\dagger}_+ \ , \\ \Phi_- &= \eta^1_+ \otimes \eta^{2\dagger}_- \ . \end{aligned} \tag{D.1.22}$$

Ils peuvent être vus comme des polyformes via l'identité de Fierz

$$\eta^1_+ \otimes \eta^{2\dagger}_\pm = \frac{1}{8} \sum_{k=0}^{6} \frac{1}{k!} \left(\eta^{2\dagger}_\pm \gamma_{\mu_k \ldots \mu_1} \eta^1_+ \right) \gamma^{\mu_1 \ldots \mu_k} \ . \tag{D.1.23}$$

Les expressions explicites des deux spineurs purs dépendent de la forme des spineurs η^1 et η^2. Nous choisissons de les paramétrer ainsi:

$$\begin{aligned} \eta^1_+ &= a \eta_+ \ , \\ \eta^2_+ &= b (k_{||} \eta_+ + k_\perp \frac{z \eta_-}{2}) \ . \end{aligned} \tag{D.1.24}$$

η_+ et $\chi_+ = \frac{1}{2} z \eta_-$ dans (D.1.24) définissent une structure $SU(2)$ comme vu précédemment. $k_{||}$ est réel et $0 \leq k_{||} \leq 1$, $k_\perp = \sqrt{1 - k_{||}^2}$. a et b sont des nombres complexes non-nuls reliés à la norme des spineurs η^i_+ par

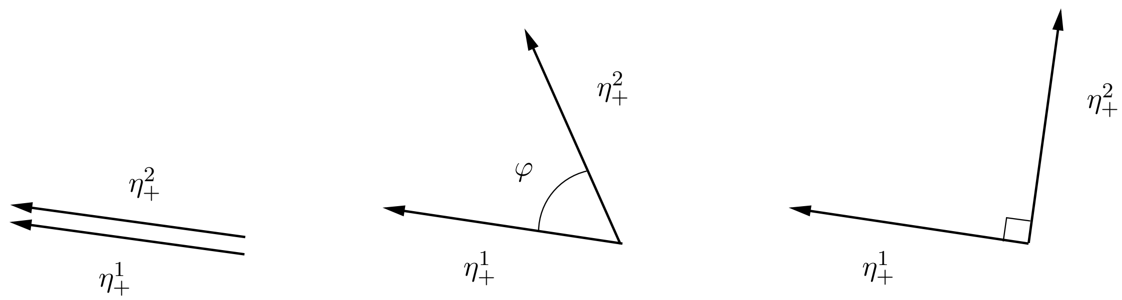
$$a = ||\eta^1_+|| \ e^{i\alpha} \ , \qquad b = ||\eta^2_+|| \ e^{i\beta} \ . \tag{D.1.25}$$

Dans la suite, on prendra toujours $|a| = |b|$, de sorte à ce que $||\eta^1_+|| = ||\eta^2_+||$. Cette condition est imposée par la présence de sources supersymétriques, ou encore par les conditions de supersymétrie en présence de sources [30, 29].

Selon les valeurs des paramètres $k_{||}$ et $k_\perp$, on peut définir différentes G-structures sur la variété interne. $k_{||}$ et $k_\perp$ peuvent être reliés à l' "angle" entre les spineurs. On peut introduire l'angle φ

$$k_{||} = \cos(\varphi), \ k_\perp = \sin(\varphi), \ 0 \leq \varphi \leq \frac{\pi}{2} \ . \tag{D.1.26}$$

Pour $k_\perp = 0$, les spineurs deviennent parallèles, donc il n'y a qu'un spineur globalement défini, et cela nous donne une structure $SU(3)$. Quand $k_\perp \neq 0$, les deux spineurs sont généralement indépendants, donc on obtient une structure $SU(2)$ [31]. Dans ce cas, on doit distinguer les deux situations $k_{||} = 0$ et $k_{||} \neq 0$. Nous les nommons respectivement une structure $SU(2)$ orthogonale et une structure $SU(2)$ intermédiaire, en référence à l'angle entre les spineurs. On a schématiquement



Structure $SU(3)$: $k_{||} = 1,\ k_\perp = 0$ — Structure $SU(2)$ intermédiaire: $k_{||} \neq 0,\ k_\perp \neq 0$ — Structure $SU(2)$ orthogonale: $k_{||} = 0,\ k_\perp = 1$

Etant donné la paramétrisation (D.1.24) des spineurs internes, on peut obtenir les expressions explicites des spineurs purs en tant que polyformes [32]

$$\begin{aligned} \Phi_+ &= \frac{|a|^2}{8} e^{i\theta_+} e^{\frac{1}{2} z\wedge\bar{z}} (k_{||} e^{-ij} - i k_\perp \omega) \ , \\ \Phi_- &= -\frac{|a|^2}{8} e^{i\theta_-} z \wedge (k_\perp e^{-ij} + i k_{||} \omega) \ , \end{aligned} \tag{D.1.27}$$

où les formes apparaissant on été définies précédemment, et les phases $\theta_\pm$ sont reliées aux phases des spineurs η^i: $\theta_+ = \alpha - \beta$, $\theta_- = \alpha + \beta$.

Un spineur pur peut toujours s'écrire sous la forme $\omega_k \wedge e^{\tilde{b} + i\tilde{\omega}}$ où ω_k est une k-forme holomorphe, et $\tilde{b}$ et $\tilde{\omega}$ sont des deux-formes réelles [7]. Le degré k de ω_k est nommé le type du spineur pur. Pour des structures $SU(2)$ intermédiaires, où $k_{||}$ et $k_\perp$ ne sont pas zéro, il est possible d'exponentier ω, et obtient par (D.1.27)

$$\begin{aligned} \Phi_+ &= \frac{|a|^2}{8} e^{i\theta_+} k_{||}\ e^{\frac{1}{2} z\wedge\bar{z} - ij - i\frac{k_\perp}{k_{||}}\omega} \ , \\ \Phi_- &= -\frac{|a|^2}{8} e^{i\theta_-} k_\perp\ z \wedge e^{-ij + i\frac{k_{||}}{k_\perp}\omega} \ , \end{aligned} \tag{D.1.28}$$

de sorte à ce que les spineurs aient un type défini: 0 et 1. Dans le cas de structure $SU(3)$ ($k_\perp = 0$), on obtient des spineurs purs de type 0 et 3

$$\Phi_+ = \frac{|a|^2}{8} e^{i\theta_+} e^{-iJ} \ , \qquad \Phi_- = -i e^{i\theta_-} \frac{|a|^2}{8} \Omega \ , \tag{D.1.29}$$

tandis que dans le cas de structure $SU(2)$ orthogonale ($k_{||} = 0$), les types sont 1 et 2:

$$\Phi_+ = -i \frac{|a|^2}{8} e^{i\theta_+} \omega \wedge e^{\frac{1}{2} z\wedge\bar{z}} \ , \qquad \Phi_- = -\frac{|a|^2}{8} e^{i\theta_-} z \wedge e^{-ij} \ . \tag{D.1.30}$$

Si un spineur pur est fermé, son type k sert à caractériser la géométrie. La variété admet alors une structure complexe le long de $2k$ directions réelles, et une structure symplectique le long des directions restantes.

Les spineurs $\Phi_\pm$ en tant que polyformes sont clairement reliés aux formes différentielles définissant les G-structures. Tout comme ces formes doivent satisfaire des conditions de structure, les spineurs purs doivent satisfaire des conditions dites de compatibilité, sur lesquelles nous ne nous étendrons pas. Dès lors, ils définiront une structure $SU(3) \times SU(3)$ sur $TM \oplus T^*M$. C'est le cas des bispineurs $\Phi_\pm$ (D.1.22). Selon la relation entre les spineurs $\eta_+^{1,2}$, cette structure se traduira sur TM en une structure $SU(3)$, $SU(2)$ orthogonale ou $SU(2)$ intermédiaire. Donc le formalisme de Géométrie Complexe Généralisé donne une contrainte topologique unifiée sur la variété M: pour avoir un vide $\mathcal{N} = 1$, on doit pouvoir trouver sur $TM \oplus T^*M$ une structure $SU(3) \times SU(3)$, ou de manière équivalente une paire de spineurs purs compatibles.

Nous allons voir à présent que nous pouvons également exprimer la contrainte différentielle en terme de ces spineurs purs.

Conditions de supersymétrie

Les conditions de supersymétrie sont données en supergravité de type II par l'annihilation des variations fermioniques (D.1.8) et (D.1.9). D'après la décomposition des paramètres de supersymétrie dix-dimensionnels (D.1.10) et (D.1.11), en facteurs quatre et six-dimensionnels, on peut séparer les variations supersymétriques en composantes externes et internes. Il est montré dans [4, 5] que ce système d'équations peut se réécrire comme un ensemble de conditions différentielles sur la paire (D.1.22) de spineurs purs compatibles:

$$(\mathrm{d} - H\wedge)(e^{2A-\phi}\Phi_1) = 0 \ , \tag{D.1.31}$$

$$(\mathrm{d} - H\wedge)(e^{A-\phi}\,\mathrm{Re}(\Phi_2)) = 0 \ , \tag{D.1.32}$$

$$(\mathrm{d} - H\wedge)(e^{3A-\phi}\,\mathrm{Im}(\Phi_2)) = \frac{|a|^2}{8} e^{3A} * \lambda(F) \ , \tag{D.1.33}$$

où λ a été défini en (D.1.3) et avec

$$\Phi_1 = \Phi_\pm \ , \qquad \Phi_2 = \Phi_\mp \ , \tag{D.1.34}$$

pour IIA/B (haut/bas). Nous prendrons par la suite $|a|^2 = e^A$. Ces conditions de supersymétrie généralisent la condition de Calabi-Yau pour les compactifications sans flux. En effet, la première de ces équations impliquent que l'un des spineurs purs (celui avec la même parité que les champs RR) doit être fermé (plus précisément de manière conforme à cause du facteur, et twisté à cause de $-H\wedge$). Une variété admettant un tel spineur pur est un Calabi-Yau Généralisé (twisté). Nous chercherons donc des solutions sur de telles variétés.

Nous rappelons que les conditions de supersymétrie et les identités de Bianchi impliquent ensemble que les équations du mouvement sont automatiquement satisfaites (pour un vide sur Minkowski).

D.2 Principaux résultats de la thèse

Jusqu'à présent, nous avons motivé la recherche de solutions supersymétriques de la supergravité dix-dimensionnelle de type II, et nous avons montré comment la Géométrie Complexe Généralisée fournit un formalisme utile pour l'étude de vides $\mathcal{N} = 1$ en présence de flux. Voici à présent la structure du reste de la thèse, et les principaux résultats, qui seront détaillés dans les sections qui suivent.

Nous discutons tout d'abord les solutions supersymétriques dix-dimensionnelles vers Minkowski, où la variété interne est une variété résoluble (tore twisté). Ces variétés sont des candidates intéressantes pour trouver de telles solutions (il a été prouvé que certaines sous-classes de ces variétés sont des Calabi-Yau Généralisés), et leurs propriétés permettent d'envisager une résolution explicite des équations de supersymétrie pour les spineurs purs et des identités de Bianchi pour les flux. La thèse fournit une revue de leurs propriétés géométriques. Puis la méthode de résolution pour trouver des solutions est présentée, et une liste de solutions connues sur ces variétés est fournie. Nous nous focalisons ensuite sur un type particulier de solutions: celles qui admettent une structure $SU(2)$ intermédiaire, définie précédemment. Afin de trouver de telles solutions, la méthode présentée doit être légèrement adaptée. Nous introduisons une base de formes particulières qui simplifie les conditions de projection de l'orientifold, et les conditions de supersymétrie. Ainsi, nous parvenons à trouver trois solutions de ce type, que nous présentons. En prenant certaines limites sur l'angle entre les spineurs internes dans ces solutions, nous pouvons retrouver des solutions connues avec structure $SU(3)$ ou $SU(2)$ orthogonale, et nous trouvons aussi une nouvelle solution. Finalement, nous dérivons les conditions pour qu'une solution à structure $SU(2)$ intermédiaire soit issue d'une transformation β d'une solution à structure $SU(3)$. Nous montrons que c'est le cas pour l'une de nos solutions.

Par la suite, nous étudions un type particulier de transformation $O(d,d)$ nommée le twist, qui peut être utilisée pour générer de nouvelles solutions supersymétriques sur Minkowski. L'idée consiste tout d'abord en une transformation qui construit les un-formes d'une variété résoluble à partir de celles d'un tore. Cette transformation est alors plongée, puis étendue, en Géométrie Complexe Généralisée dans une transformation $O(d,d)$ locale, afin de relier des solutions sur un tore à des solutions sur des variétés nilpotentes ou résolubles. Les conditions pour générer, à l'aide de cette transformation, des solutions supersymétriques, sont discutées, et utilisées pour retrouver toutes les solutions connues sur variétés nilpotentes, et trouver une nouvelle solution sur une variété résoluble. Nous présentons également une nouvelle solution complètement localisée sur une variété résoluble, et discutons par ailleurs la possibilité d'obtenir des solutions non-géométriques à l'aide du twist. Finalement, nous réécrivons les conditions de supersymétrie en corde hétérotique en termes de spineurs purs, puis nous les utilisons pour relier à l'aide du twist deux solutions avec flux connues dans ce contexte.

Finalement, nous discutons la possibilité d'obtenir des solutions non-supersymétriques sur un espace-temps quatre-dimensionnel de de Sitter. La motivation pour de telles solutions est cosmologique. Nous expliquons tout d'abord les difficultés majeures rencontrées lorsque l'on essaye d'obtenir des solutions de supergravité avec constante cosmologique Λ positive. Puis nous proposons un ansatz pour des sources brisant la supersymétrie, qui pourrait aider à augmenter la valeur de Λ. Cette ansatz se base sur l'idée de préserver un certain ensemble d'équations du premier ordre basé sur la structure $SU(3)$, ce malgré la brisure des conditions de supersymétrie. Puis nous donnons un exemple explicite de solution de Sitter avec cet ansatz, où la variété interne est résoluble. Cette solution peut être comprise comme une déviation de la nouvelle solution supersymétrique trouvée auparavant grâce au twist. Enfin, nous fournissons une analyse partielle de la stabilité quatre-dimensionnelle de la solution trouvée.

La thèse se termine par une conclusion qui résume le travail effectué et propose des idées et des directions à suivre pour la suite.

D.2.1 Solutions sur variétés résolubles

Nous avons motivé l'utilisation du formalisme de la Géométrie Complexe Généralisée pour étudier les vides supersymétriques de supergravité de type II avec flux non-triviaux, lorsque l'espace-temps dix-dimensionnel est séparé en l'espace-temps de Minkowski quatre-dimensionnel et une variété interne M six-dimensionnelle. En particulier, pour obtenir des vides $\mathcal{N} = 1$ sur Minkowski avec flux non-triviaux, l'espace interne est caractérisé comme étant une variété de Calabi-Yau Généralisé.

Ici, nous étudions la possibilité d'obtenir des exemples explicites de solutions avec flux non-triviaux sur Calabi-Yau Généralisé. Les exemples les plus simples de vides avec flux non-triviaux sont des Calabi-Yau conformes en type IIB (dans le cas le plus simple un T^6 conforme), avec un $O3$ et un flux donné par une trois-forme auto-duale. Une méthode standard [33] pour produire de nouveaux vides avec flux non-triviaux est de T-dualiser les solutions sur Calabi-Yau conformes. Les variétés résultantes sont des tores twistés, c'est-à-dire des fibrations de cercles sur une base donnée par un tore. Mathématiquement parlant, ceux sont des variétés résolubles: ces variétés sont construites à partir de groupes de Lie particuliers nommés les groupes résolubles. Un sous-ensemble de ces groupes est constitué de groupes dit nilpotents, à partir desquels on peut construire les variétés nilpotentes. Il a été montré que les variétés nilpotentes sont toutes des Calabi-Yau Généralisés [23]. Et en effet, certains vides supersymétriques avec flux non-triviaux ont été trouvés sur ces variétés via la T-dualité.

En utilisant la Géométrie Complexe Généralisée, au lieu des dualités, on peut essayer de trouver des vides sur un Calabi-Yau Généralisé (par exemple l'une des variétés nilpotentes) en résolvant directement les contraintes de supersymétrie, et les identités de Bianchi pour les flux. Cette stratégie a été utilisée dans [29] pour déterminer les vides avec flux sur les variétés nilpotentes et quelques variétés résolubles. Les auteurs ont retrouvé quelques solutions connues, qui avaient été obtenues précédemment par T-dualité en partant d'une solution sur un Calabi-Yau conforme, mais ils ont aussi trouvé de nouveaux vides non T-duaux. Ces solutions ont été obtenues en réalisant une recherche exhaustive sur toutes les variétés nilpotentes six-dimensionnelles, et sur quelques variétés résolubles.

Une telle recherche a été possible car la méthode de résolution sur ces variétés est plutôt algorithmique. En particulier les variétés considérées sont parallélisables donc elles possèdent, via leur formes de Maurer-Cartan, une base six-dimensionnelle de un-formes globalement définies. Grâce à cette base, on peut construire des spineurs purs généraux, et essayer d'ajuster leurs paramètres libres pour obtenir une solution.

Dans la thèse, nous donnons tout d'abord une revue des propriétés géométriques et algébriques des variétés nilpotentes et résolubles. Nous donnons ensuite plus de détails sur la méthode de résolution et sur les solutions trouvées de cette manière. La suite de l'étude est dédiée à un type de solutions particulières, celles qui admettent une structure $SU(2)$ intermédiaire, définie précédemment. Afin de trouver de telles solutions, on doit adapter légèrement la méthode de résolution. En particulier, on introduit certaines variables qui simplifie les conditions de projection de l'orientifold, et les conditions de supersymétrie. On parvient alors à trouver trois nouvelles solutions non T-duales à un Calabi-Yau conforme, que l'on présente ensuite. On discute enfin certaines relations qu'elles ont avec des solutions à structure $SU(3)$ ou $SU(2)$ orthogonale.

D.2.2 Transformation de twist en corde de type II et hétérotique

Précédemment, nous avons discuté des exemples de solutions Minkowski supersymétriques avec flux non-triviaux sur variété résoluble. Nous avons également fourni une liste de solutions connues.

Les solutions trouvées sur variétés résolubles non-nilpotentes ne sont pas T-duales à des configurations sur T^6 conformes. Parmi les solutions trouvées sur les variétés nilpotentes, seules celles correspondant à l'algèbre n 3.14 sont aussi non T-duales à un T^6 conforme. De plus, en type IIB, en partant d'un T^6 avec un $O3$ et un champ B non-trivial, et en réalisant deux T-dualités indépendantes, on aboutit dans la même théorie à une variété nilpotente avec premier nombre de Betti étant égal à 5 ou 4 [33, 52]. n 3.14 au contraire a son premier nombre de Betti $b_1(M) = 3$. Par conséquent, on peut s'interroger si toutes ces solutions, T-duales ou pas, peuvent être reliées par une transformation plus générale. On présente ici une telle transformation que nous nommons le twist.

L'idée consiste tout d'abord à construire un opérateur $GL(d)$ qui transforme la base de un-formes du tore en la base de un-formes de Maurer-Cartan d'une variété résoluble donnée. Les un-formes de Maurer-Cartan reflètent la topologie de la variété. Par conséquent, un tel opérateur devrait encoder la topologie de la variété résoluble atteinte par la transformation. D'une certaine manière, la matrice $\mu(t)$ des variétés résolubles quasi-abéliennes est déjà un exemple d'opérateur $GL(d)$ encodant la topologie d'une variété. Elle encode la fibration du fibré de Mostow. Pour ces variétés, notre opérateur sera en fait très proche de ces matrices $\mu(t)$.

Si l'on parvient à relier les formes du tore à celles des variétés résolubles, il est tentant d'essayer de relier des solutions complètes. Dans le formalisme de Géométrie Complexe Généralisée, une manière naturelle de transformer une solution est d'agir sur ces spineurs purs avec un élément de $O(d, d)$. Un exemple bien connu est l'action du groupe de T-dualité $O(n, n)$ (voir [53] pour une revue sur le sujet) sur une variété ayant n isométries. Il est donc naturel de plonger la transformation $GL(d)$ responsable du changement de topologie dans une transformation $O(d, d)$ agissant de manière équivalente sur les formes et les vecteurs du fibré tangent généralisé. Etant donné ce plongement, on peut étendre la transformation en incluant d'autres ingrédients. En particulier, ceux-ci permettront de transformer le champ B avec une transformation dite B, transformer la métrique avec un facteur d'échelle, et le dilaton sera changé en conséquence. On peut également inclure une paire de transformations $U(1)$

agissant sur les spineurs purs par multiplication avec des phases. De la sorte, on peut transformer tout le secteur NSNS à notre guise. L'action $O(d,d)$ étant appliquée sur le fibré tangent généralisé, les flux RR ne sont pas transformés directement, mais au travers des spineurs purs et des équations de supersymétrie les définissant (D.1.33). Notez que la transformation générale que l'on peut déduire pour les flux RR mélange les secteurs NSNS et RR, ce qui n'est pas le cas de la T-dualité. Une manière de réaliser ce genre de mélange est d'utiliser la U-dualité, mais notre transformation n'a pas l'air d'y être reliée.

Nous proposons donc d'utiliser la transformation de twist pour relier les solutions sur le tore aux solutions sur les variétés résolubles. A la différence de la T-dualité, le twist est une transformation $O(d,d)$ locale. Par conséquent, alors que la première est une symétrie des équations du mouvement, la deuxième ne l'est pas en général. Néanmoins, des contraintes générales sur le twist peuvent être formulées de sorte à ce qu'il préserve les conditions de supersymétrie. Dans certains cas, ces contraintes sont suffisamment simples pour être résolues. On peut alors utiliser la transformation comme une technique pour générer des solutions. Par exemple, nous sommes capables de relier toutes les solutions en type IIB présentées précédemment sur les variétés nilpotentes, y compris la solution non T-duale sur n 3.14 qui semblait isolée. Pour les variétés résolubles, on peut également retrouver la solution non T-duale sur s 2.5 en type IIB. On utilise également le twist dans la thèse pour construire une nouvelle solution sur une nouvelle variété résoluble. Enfin, on discute également la possibilité d'obtenir des solutions non-géométriques.

Les transformations de twist peuvent aussi être appliquées dans le contexte de la corde hétérotique, pour relier deux solutions supersymétriques sur des variétés ayant des topologies différentes. Ces solutions ont été reliées auparavant par ce qui est connu sous le nom de transition Kähler/non-Kähler: la relation est établie via une chaîne compliquée et indirecte de dualités impliquant un passage en théorie M [54, 55, 56, 57, 58, 59, 60, 61, 62]. Afin de relier ces solutions par le twist, nous discutons au préalable la reformulation des conditions de supersymétrie en corde hétérotique, en terme de Géométrie Complexe Généralisée.

Dans l'appendice associé, certains points sont abordés plus en détails. Nous donnons tout d'abord une construction plus détaillée des un-formes des variétés résolubles, et donnons une liste de ces variétés en termes de ces un-formes globalement définies. Puis nous discutons les solutions potentiellement non-géométriques T-duales aux solutions sur variétés résolubles. Enfin, dans le contexte de la corde hétérotique, on étend le fibré tangent généralisé pour inclure le fibré de jauge, afin de transformer les champs de jauges directement par un sous-ensemble de transformations locales de $O(d+16, d+16)$.

D.2.3 Sources brisant la supersymétrie et les vides de de Sitter

Récemment, de nombreux travaux en compactifications de cordes se sont focalisés sur la recherche de solutions de de Sitter. Ce regain d'intérêt est dû à de récentes données cosmologiques suggérant que nous vivons dans un univers en expansion caractérisé par une constante cosmologique faible, mais positive.

Les solutions de de Sitter sont bien plus difficiles à trouver que celles sur Minkowski ou Anti de Sitter. Tout d'abord, l'espace-temps de de Sitter n'est pas compatible avec la supersymétrie. Comme indiqué précédemment, les conditions de supersymétrie et les identités de Bianchi pour les flux impliquent que l'ensemble des équations du mouvement soient satisfaites. Ainsi, la supersymétrie permet une simplification technique conséquente dans la recherche des solutions, puisque l'on est amené à résoudre des équations du premier ordre plutôt que des équations du second ordre.

Une deuxième difficulté concerne l'obtention d'une constante cosmologique positive Λ. Comme nous allons le voir plus en détails, pour des solutions de supergravité, avoir $\Lambda > 0$ requiert un ajustement non-trivial des paramètres géométriques et des flux de la solution.

Finalement, étant donné une solution dix-dimensionnelle, on doit vérifier que sa réduction quatre-dimensionnelle soit stable, c'est-à-dire que les extrema correspondant du potentiel quatre-dimensionnel doivent être des minima (pour des modèles d'inflation dit de “slow roll”, on peut revenir légèrement sur ce point). Cette contrainte est également difficile à satisfaire, et jusqu'à présent, aucune solution de de Sitter stable admettant seulement des ingrédients dix-dimensionnels classiques n'a été trouvée.

Ici, nous nous intéressons aux solutions de de Sitter en supergravité de type IIA. Dans ce contexte, il existe plusieurs théorèmes no-go allant contre l'existence de vides de de Sitter, et des manières de les contourner ont également été proposées [12, 69, 70, 71, 42, 72, 73, 74, 75]. Par conséquent, l'obtention de vides de de Sitter requiert certaines conditions nécessaires (mais non suffisantes). Tout d'abord, des orientifolds sont nécessaires, tout comme pour les compactifications vers Minkowski [12]. Nous considérerons en particulier des sources O6/D6. Dans ce cas, la variété interne doit avoir un rayon de courbure négatif et une masse de Roman non-nulle [12, 70, 42, 73]. Une autre possibilité est de permettre des flux non-géométriques, mais nous ne suivrons pas cette approche ici.

Nous considérerons donc des configurations de type IIA avec une trois-forme NSNS et une zéro- et deux-formes RR. Nous considérerons également des sources qui remplissent l'espace-temps quatre-dimensionnel et seulement de dimension $p = 6$. Comme les sources pourraient avoir une intersection, nous considérerons un dilaton constant, $e^{\phi} = g_s$, et un facteur conforme constant.

De plus, en supposant que les sources sont supersymétriques, on peut combiner les traces quatre- et six-dimensionnelles des équations d'Einstein, et l'équation du mouvement du dilaton pour obtenir

$$R_4 = \frac{2}{3}(g_s^2|F_0|^2 - |H|^2) \ , \tag{D.2.1}$$

$$R_6 + \frac{1}{2}g_s^2|F_2|^2 + \frac{3}{2}(g_s^2|F_0|^2 - |H|^2) = 0 \, . \tag{D.2.2}$$

La seconde équation n'est qu'une contrainte sur les quantités internes, tandis que la première fixe R_4. On retrouve via ces deux équations les contraintes minimales: avoir $F_0 \neq 0$ et $R_6 < 0$. La contribution négative de H n'est pas toujours facile à contrebalancer, car F_0 et H ne sont pas indépendants (ils sont reliés via l'équation du mouvement de H et l'identité de Bianchi de F_2). Ajouter des flux comme F_4 et F_6 n'aide aucunement car ils contribuent avec des signes négatifs. Par conséquent, en pratique, F_0 n'est souvent pas suffisant pour obtenir un vide de de Sitter. C'est la raison pour laquelle jusqu'à présent, tous les exemples connus de vides de de Sitter stables requièrent des ingrédients additionnels comme des monopoles KK et des lignes de Wilson [71], des flux non-géométriques [76], ou des corrections α' et des $D6$-branes sondes [77].

Ici, nous aimerions voir si, en revenant sur une hypothèse, il serait possible de trouver des solutions de de Sitter dans des compactifications géométriques classiques. Nous décidons de revenir sur l'hypothèse de sources préservant la supersymétrie. Comme nous nous intéressons à des solutions non-supersymétriques, il n'y a a priori pas de justification pour préserver la supersymétrie des sources, si ce n'est leur stabilité sur laquelle nous reviendrons. Par conséquent nous proposons un ansatz pour les sources brisant la supersymétrie. Ceci donnera une nouvelle contribution positive à R_4.

Pour une source supersymétrique, on peut remplacer la forme de volume sur le volume d'univers de la brane par le pullback du spineur pur non-intégrable [45, 30]

$$\left(i^*[\mathrm{Im}\,\Phi_-] \wedge e^{\mathcal{F}}\right) = \frac{|a|^2}{8}\sqrt{|i^*[g] + \mathcal{F}|}\mathrm{d}^{\Sigma}x \, , \tag{D.2.3}$$

où i dénote le plongement du volume d'univers dans la variété interne M, g est la métrique interne et $\mathcal{F}$ le tenseur de Faraday associé au champ de jauge sur le volume d'univers de la brane. Afin de considérer des sources non-supersymétriques, nous proposons de modifier (D.2.3) en

$$\left(i^*[\mathrm{Im}\,X_-] \wedge e^{\mathcal{F}}\right) = \sqrt{|i^*[g] + \mathcal{F}|}\mathrm{d}^{\Sigma}x \, , \tag{D.2.4}$$

où X_- est une polyforme impaire non-pure, pulled back depuis l'espace entier. X_- est une expansion générale sur la base de $TM \oplus T^*M$ donnée par les spineurs purs $\Phi_\pm$. Pour les configurations supersymétriques, X_- se réduit à $8\Phi_-$. Le nouveau terme de source (D.2.4) permet de réécrire le scalaire de Ricci quatre-dimensionnel (D.2.1) comme

$$R_4 = \frac{2}{3}\left(\frac{g_s}{2}(T_0 - T) + g_s^2|F_0|^2 - |H|^2\right) , \tag{D.2.5}$$

où T est la trace du tenseur d'énergie-impulsion, et T_0 est la partie supersymétrique de la trace: pour les sources supersymétriques, $T_0 = T$. On peut montrer que $T_0 > 0$, ce qui donne une contribution positive à R_4.

L'expression (D.2.4) aide de plus à résoudre les équations d'Einstein internes avec des flux non-supersymétriques. En effet, nous sommes capables de trouver un exemple concret de solution de de Sitter dix-dimensionnelle en supergravité de type IIA.

Afin de mieux justifier cet ansatz pour les sources brisant la supersymétrie, on montre que (avec $\mathrm{d}_H = \mathrm{d} - H\wedge$)

$$\begin{aligned} &\mathrm{d}_H(e^{2A-\phi}\,\mathrm{Re}\,X_-) = 0\,, \\ &\mathrm{d}_H(e^{4A-\phi}\,\mathrm{Im}\,X_-) = c_0 e^{4A} * \lambda(F)\,, \end{aligned} \tag{D.2.6}$$

où c_0 est une constante fixée par les paramètres de la solution. Ce sont des équations du premier ordre qui généralisent les conditions de supersymétrie (D.1.32) et (D.1.33) sur Φ_-. Notez que comme dans le cas supersymétrique, la seconde équation dans (D.2.6) implique que les équations du mouvement des flux RR sont automatiquement satisfaites, à condition qu'il n'y a pas de source NSNS ($\mathrm{d}H = 0$). En effet, en différenciant (D.2.6), on obtient automatiquement

$$(\mathrm{d} + H\wedge)(e^{4A} * F) = 0\ . \tag{D.2.7}$$

Notez que la présence de c_0, génériquement non égale à un, indique que, à la différence des calibrations généralisées [45, 30, 46, 11], la densité d'énergie de source n'est ici pas minimisée. Nous cherchons ici les extrema de la combinaison de l'énergie de la brane et du bulk.

L'idée de résoudre des équations du premier ordre pour trouver des solutions non-supersymétriques n'est pas nouvelle. Une généralisation des équations sur les spineurs purs (D.1.31), (D.1.32) et (D.1.33) pour étudier les solutions non-supersymétriques a été proposée récemment [78]. L'idée consiste à exprimer la violation des conditions de supersymétrie comme une expansion sur la base $Spin(6,6)$ construite à partir des spineurs purs. Par exemple, pour les compactifications vers Minkowski, les équations du premier ordre modifiées sont

$$\begin{aligned} &\mathrm{d}_H(e^{2A-\phi}\Phi_1) = \Upsilon\,, \\ &\mathrm{d}_H(e^{A-\phi}\,\mathrm{Re}\,\Phi_2) = \mathrm{Re}\,\Xi\,, \\ &\mathrm{d}_H(e^{3A-\phi}\,\mathrm{Im}\,\Phi_2) - \frac{|a|^2}{8}e^{3A} * \lambda(F) = \mathrm{Im}\,\Xi\,, \end{aligned} \tag{D.2.8}$$

où schématiquement

$$\begin{aligned} \Upsilon \;&=\; a_0\Phi_2 + \tilde{a}_0\overline{\Phi}_2 + a^1_m\gamma^m\Phi_1 + a^2_m\Phi_1\gamma^m + \tilde{a}^1_m\gamma^m\overline{\Phi}_1 + \tilde{a}^2_m\overline{\Phi}_1\gamma^m \\ &\quad + a_{mn}\gamma^m\Phi_2\gamma^n + \tilde{a}_{mn}\gamma^n\overline{\Phi}_2\gamma^m\,, \end{aligned} \tag{D.2.9}$$

$$\Xi \;=\; b_0\,\Phi_1 + \tilde{b}_0\,\overline{\Phi}_1 + b^1_m\gamma^m\Phi_2 + b^2_m\Phi_2\gamma^m + b_{mn}\gamma^m\Phi_1\gamma^n + \tilde{b}_{mn}\gamma^n\overline{\Phi}_1\gamma^m\,. \tag{D.2.10}$$

Dans le cas particulier d'une structure $SU(3)$, cette décomposition est équivalente à une expansion sur classes de torsion $SU(3)$. Cette idée a été utilisée pour chercher des solutions non-supersymétriques sur Minkowski et Anti de Sitter [78, 72, 79, 80, 75]. Cependant, cette approche suppose que la supersymétrie quatre-dimensionnelle n'est pas brisée explicitement, et que la brisure n'apparaît que sur la variété interne. Pour cette raison, cela ne s'applique pas directement aux compactifications de de

Sitter.

Notre solution de de Sitter explicite a été trouvée sur la variété résoluble $\mathfrak{g}_{5.17}^{p,-p,\pm 1} \times S^1$ d'algèbre $(q_1(p25+35), q_2(p15+45), q_2(p45-15), q_1(p35-25), 0, 0)$. Pour $p = 0$, cette algèbre se réduit à s 2.5 (une algèbre résoluble sur laquelle on connaît déjà des solutions supersymétriques), tandis que pour $p \neq 0$, la variété admet une solution supersymétrique à condition qu'une certaine combinaison de paramètres, que l'on nomme λ, est égale à un. Pour un λ générique, les équations de spineurs purs ne sont pas satisfaites et la supersymétrie est brisée. Cette configuration sert d'ansatz pour trouver une solution de de Sitter. Notez qu'il est pratique d'avoir une limite supersymétrique dans laquelle notre construction peut être testée.

La thèse contient une présentation plus détaillée du traitement des sources brisant la supersymétrie et la forme explicite de la solution de de Sitter. De plus, nous déterminons puis étudions le potentiel effectif quatre-dimensionnel. En particulier, nous discutons comment les sources non-supersymétriques contribuent à de nouveaux termes dans le potentiel. Nous analysons également la stabilité de la solution en terme du volume et du dilaton. La stabilité des autres moduli reste indéterminée. De même, la question de savoir si les propositions (D.2.4) et (D.2.6) peuvent fournir des sources stables n'est pas traitée, et nous espérons y revenir plus tard.

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